A STUDY OF A CLASS OF POLYNOMIALS DEFINED BY GENERALISED RODRIGUE'S FORMULA

A THESIS

PRESENTED BY

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CERTIFICATE

This is to certify that the work embodied in the thesis entitled " A CLASS OF POLYNOMIALS DEFINED BY GENERALISED RODRIGUE'S FORMULA " being submitted by Sanjay Khanna, to fulfill the partial requirement for the degree of M. Phil.of Bundelkhand University Jhansi, U.P. is upto the mark, both in academic contents and quality of presentation. I further certify that this work has been done by him under my supervision and guidance.

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PREFACE

The present work is outcome of the studies done by me in the field of Special Functions, with special emphasis on " Class of polynomials by generalised Rodrigue's formula " at the department of Mathematics and Statistics, Bundelkhand University, Jhansi during the course of studies for the degree of M. Phil.

The present work has been done under the able guidance of Dr. P.N. Shrivastava, Reader and Head of the Mathematics Department, Bundelkhand University, Jhansi.

I express my deepest sense of gratitute to Dr. P.N. Shrivastava for competent guidance and unbounding interest in the preparation of this thesis. I am also thankful to Dr. V.K. Sehgal and Dr. V.K.Singh of the Department for their continuous encouragement.

This thesis consists of three Chapters each divided into several sections (Progressively 1.1, 1.2,........). The formulae are numbered progressively within each section. For instance (3.3.8) denotes the 8th formula in 3rd article of 3rd Chapter. References are given at the end of the thesis in alphabatical order.

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CONTENTS

CHAPTER-I

INTRODUCTION

The purpose of thâ present Chapter is to give a brief historical accounts of the work done in the field of unification of classical polynomials defined by generalized Rodrigue formula. The classical polynomials were considered the solutions of partial differential equation governing the behaviour of certain physical quantities like wave equation, Laplace equation, diffusion equation etc. and have been studied by many authors in their own ways. Prof. Bateman (1882-46) is considered as one of the greatest authorities who studied the subject in a classical manner in the name of special functions.

Apart from its usefulness by Scientists, it is found that the subject is more interesting while dealing with certain theoretical problems. The chief organs in the study of special functions are Rodrigue's type formulae, generating relations, recurrence relations, operational formulae etc.

A great amount of work has been done on the study of classical polynomials like Hermite, Laguerre, Bessel, Legendre, Gegenbauer, Bell, Truesdell and Humbert Polynomials. So in the section below it is proposed to give a brief history of polynomials defined by Rodrigue's type formula.

1.1 RODRIGUE'S FORMULA AND ITS GENERALIZATIONS.

The classical orthogonal polynomials have a generalized Rodrigue's formula of the form.

$$(1.1.1)\quad F_n(x) = \frac{1}{K_n W(x)} D^n [W(x).X^n],$$

$$D \equiv \frac{d}{dx}. \quad n = 0, 1, 2 \cdots$$

where K_n is a constant, x isa polynomial in x whose coefficients are independent of n, W(x) is the weight function and $F_n(x)$ is a polynomial of degree n in x.

The Rodrigue's formula for Legendre, Laguerre and Hermite polynomials which satisfy (1.1.1) are the particular cases of the Rodrigue's formula are as follows:-

$$(1.1.2)\quad P_n(x) = \frac{1}{2^n n!} D^n (x^2-1)^n .$$

$$(1.1.3)\quad L_n(x) = \frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x}) .$$

$$(1.1.4)\quad H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2}) .$$

The Rodrigue's formula for ultraspherical polynomials $P_n^{\lambda}(x)$ and Jacobi polynomials (1859) $P_n^{(\alpha,\beta)}(x)$ which are the generalizationsof Legendre polynomials have led to think and generalize them in different directions.

In 1901 Apell [4] considered a new generalization of ~~xx~~ the class of polynomials.

(1.1.5) $$R_{2n}(x) = D^n [x^n (1-x^2)^n] .$$

In 1933 Cioranensen [14] generalized Legendre polynomials by defining the formula

(1.1.6) $$P_n(x,Q) = \frac{1}{A_n} \frac{d^{(k-1)n}}{dx^{(k-1)n}} \left[\{Q(x)\}^n \right],$$

where $Q_n(x) = (x-a_1)(x-a_2)\ldots(x-a_k)$ is a polynomial of degree K and A_n is any suitable constant.

In 1918 Nielsen [42] derived a formula for $H_{m+n}(x)$ as

(1.1.7) $$H_{m+n}(x) = \sum_{k=0}^{\min(m,n)} (-2)^k \binom{m}{k}\binom{n}{k} k!\, H_{m-k}(x)\, H_{n-k}(x) .$$

P.K. Menon [39] in 1941 has generalized Legendre polynomials as

(1.1.8) $$P_{n,s}(z) = \frac{1}{n!\, s^n} D^n (z^n - 1)^n .$$

In 1941 Burchnall [12] defined an operational formula as,

(1.1.9) $$(D - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x)\, D^k .$$

In 1938 Angelescue [5] considered the polynomials $\pi_n(x)$ connected with Appell and defined as,

(1.1.10) $$\pi_n(x) = e^x D^n \{ e^{-x} A_n(x) \}$$

where the set of polynomials $A_n(x)$ forms an Appell set.

In an attempt to generalize the work of J.G.Steffenson [59] (1928), Maurice de Duffahel [28] (1936), L. Toscano [61] (1952), P.Humbert [30] (1923) and Chak [13] (1956) introduced two classes of polynomials and studied them separately which are given by,

(1.1.11) $G_{n,k}^{(\alpha)}(x) = x^{-\alpha-kn} e^{x} \theta^{n} (x^{\alpha} e^{-x})$,

and

(1.1.12) $P_{n,r}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{x^r} D^n [x^{n+\alpha} e^{-x^r}]$,

where $\theta = x^{k+1} D$.

Chak obtained following two generating functions for as,

(1.1.13) $\sum_{n=0}^{\infty} t^n P_{n,r}^{(\alpha-2n)}(x) = \frac{(1-\sqrt{1+4t})^{\alpha+1}}{2^{\alpha+1}\sqrt{1+4t}} \exp\left\{x^r\left[1-\left(\frac{1+\sqrt{1+4t}}{2}\right)^r\right]\right\}$

and

(1.1.14) $\sum_{n=0}^{\infty} t^n P_{n,r}^{(\alpha+n)}(x) = \left(\frac{1-\sqrt{1-4t}}{2t}\right)^{\alpha} e^{x^r} \frac{\left\{1-\left(\frac{1-\sqrt{1-4t}}{2}\right)^r\right\}}{\sqrt{1-4t}}$.

In 1949 Krall-Frink [39] obtained a class of polynomials which they called " Bessel polynomials" defined as,

(1.1.15) $y_n(x;a,b) = b^{-n} x^{2-a} e^{b/x} D^n [x^{2n+a-2} e^{-b/x}]$.

In 1948 Agarwal [1] showed that the Bessel polynomials are the limiting case of Jacobi polynomials and are given by the relation,

(1.1.16) $$y_n(x;a,b) = \underset{\theta\to\infty}{Lt} \frac{\Gamma n+1\, \Gamma\theta}{\Gamma n+\theta} P_n^{(\theta-1,\,a-\theta-1)}\left(1+\frac{2\theta x}{p}\right).$$

E.T. Bell (1934) [6] considers the polynomials $\xi(x,t;r)$ defined by

(1.1.17) $$\xi(x,t;r) = \exp(-x t^r)\, D^n(e^{xt^r}) \quad , \quad D \equiv \frac{d}{dt};$$

and called them as exponential polynomials. Bell showed that the function.

(1.1.18) $$\zeta_n(x,t;2r) = e^{-\frac{x}{2}t^{2r}}\, \xi(-x,t;2r),$$

forms an orthogonal set of polynomials in the interval $(-\infty,\infty)$ i.e.,

(1.1.19) $$\int_{-\infty}^{\infty} \zeta_n \zeta_m \, dt = 0 \quad , \qquad m \neq n .$$

He also extended Appell polynomials .

Murice de Duffahel [28] in 1936 has defined polynomials $P_n(x)$ as.

(1.1.20) $$P_n(x) = \frac{1}{n!} e^{x^2} D^n(x^n e^{-x^2}) \,.$$

In the more general form

(1.1.21) $$\frac{1}{\rho(x)} \frac{d^n}{dx^n}\left\{\rho(x)\,[X(x)]^n\right\} \quad ; (n=0,1,2 \cdots)$$

where $\rho(x)$ and $X(x)$ are independent of n, $\rho(x)$ is the infinitely differentiable function and $X(x)$ is a polynomial. Tricomi [63] showed that the degree of $X(x)$ should not exceed 2 in order that all the polynomials may be generated by (1.1.21) and may be reduced to one of the classical orthogonal polynomials by a linear change of independent

variable.

In 1956 A.M. Chak [13] considered the polynomials.

$$(1.1.22)\quad Q_{n,k}^{(\alpha)}(x) = D^n(x^{n+\alpha} e^{-x^r}).$$

In 1959 F.J. Palas [44] started with the generating function.

$$(1-t)^{-1}\exp[x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}(x)\, t^n,$$

where $u(t) = 1-(1-t)^{-k}$ and showed that the polynomial T_{kn} satisfy tne Rodrigue's formula.

$$(1.1.23)\quad T_{kn}(x) = \frac{1}{n!} e^{x^k}\left(\frac{d}{dx}\right)^n (x^n e^{-x^k}).$$

In 1958 Riordan [49] studied the Bell polynomial $H_n[g,h]$ as,

$$(1.1.24)\quad H_n[g,h] = (-1)^n e^{hg} D^n e^{hg},$$

and studied operational formulae related to (1.1.24) as

$$(D+hg')^n = (-1)^n H_n[g,h],$$

where $g' = \frac{d}{dx}g$, h is a constant and g some specified function.

In 1960 Carlitz [15] gave an analogous formula for Laguerre polynomials as,

$$(1.1.25)\quad \prod_{j=1}^{n}(xD - x + a + j) = n!\sum_{k=0}^{n}\frac{x^k}{k!} L_{n-k}^{(a+k)}(x)\, D^k;$$

a relation analogous to

$$(1.1.26)\quad (D-2x)^n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} H_{n-k}(x)\, D^k.$$

In 1962 Gould Hopper [29] gave formulas similar to (1.1.25), as

$$\prod_{j=1}^{n} (xD + a + j) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+a}{n-k} (n-k)!\, x^k D^k ; \tag{1.1.27}$$

and the two generalization of Hermite polynomials as,

$$H_n^{(r)}(x, a, p) = (-1)^n x^{-a} D^n (x^a e^{-px^r}) \tag{1.1.28}$$

and

$$g_n^{(r)}(x, h) = e^{hD^r} x^n , \qquad \text{where } D \equiv \frac{d}{dx}. \tag{1.1.29}$$

The operational formulas for these polynomials are as follows:-

$$\mathcal{S}^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}^{(r)}(x, a, p)\, D^k , \tag{1.1.30}$$

where $\mathcal{S} = D_x - prx^{r-1} + \frac{a}{x}$.

The special case of (1.1.30) when $a = 0,\ r = 2,\ p = 1$ is given by Burchnall [12]

$$(x\mathcal{S})^n = \sum_{k=0}^{n} P(x, k)\, x^k D^k , \tag{1.1.31}$$

where $P(x,k) = \sum_{j=0}^{n-k} (-1)^j \binom{j+k}{j} S(n, j+k)\, x^j H_j^{(r)}(x,a,p)$

and

$$S(n, j) = \frac{(-1)^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} k^n .$$

$S(n, j)$ are Stirling numbers of second kind.

In 1960 Raj Gopal [47] obtained an operational formula for Bessel polynomials

(1.1.32) $$x^{2n}\left[D+\frac{2(nx+1)}{x^2}\right] = \sum_{r=0}^{n}\binom{n}{r} 2^{n-r} x^{2r} y_{n-r}(x, 2+2r, 2) D^r y .$$

In 1966 S.K. Chatterjea [23] gave an operational formula as

(1.1.33) $$\prod_{j=1}^{n}\left\{x^k D+(a+kj-pr x^r)x^{k-1}\right\} y = \sum_{s=0}^{n}\binom{n}{s} s^{ks} F_{n-s}^{(r)}(x, a+ks, k, p) D^s y .$$

In 1966 S.K. Chatterjea [23] also introduced a generalized function as

(1.1.34) $$F_n^{(r)}(x; a, k, p) = x^{-a} e^{px^r} D^n\left[x^{kn+a} e^{-px^r}\right].$$

Following Gould-Hopper [29] , Singh-Srivastava [55] gave the generalization of Laguerre polynomials

(1.1.35) $$L_n^{(\alpha)}(x, r, p) = \frac{1}{n!} x^{-\alpha} e^{px^r} D^n\left[x^{\alpha+n} e^{-px^r}\right].$$

Following Palas [44] , S.K. Chatterjea [21] gave a same generalization by the following notation and obtained an operational relations as

(1.1.36) $$T_{r_n}^{(\alpha)}(x) = \frac{1}{n!} x^{\alpha} e^{-x^r} \frac{d^n}{dx^n}\left[x^{\alpha+n} e^{-x^r}\right],$$

(1.1.37) $$\prod_{j=1}^{n}(xD+\alpha+j-pkx^k) y$$
$$n! \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(n-r)}^{(\alpha+r)}(x) D^r y .$$

Following N.Obreskov [43] , in 1964 Chatterjea [20,21]

generalized Bessel polynomials as

(1.1.38) $M_n^{(k)}(x,a,b)$

$$= b^{-n} x^{k-a-(k-2)n} e^{b/x} D^n (x^{kn+a-k} e^{-b/x}).$$

and showed that

(1.1.39) $$M_n^{(k)}(x,a,b) = n! \left(-\frac{x}{b}\right)^n L_n^{(-kn-a+k+1)}(b/x).$$

C.M. Joshi and J.P. Singhal [33] introduced a class of polynomials unifying the generalized Hermite and Laguerre polynomials by means of Rodrigue's formula.

(1.1.40) $$J_n^{(\alpha)}(x,r,p,q) = C(q,n)\, x^{-\alpha} e^{px^r} D^n (x^{\alpha+nq} e^{-px^r}),$$

where $$C(q,n) = \frac{(-1)^{\frac{n}{2}(q-1)(q-2)}}{2^{\frac{n}{2}q(q-1)} (1)_{nq(2-q)}} .$$

q - being a non negative integer .

In 1967 R.P. Singh [54] generalized Truesdell polynomials as

(1.1.41) $$T^{(\alpha)}(x,r,p) = x^{-\alpha} e^{px^r} (xD)^n (x^\alpha e^{-px^r}).$$

In 1969 P.N. Shrivastava [50] considered generalized polynomials as

(1.1.42) $$G_n(h,g) = e^{-hg} (xD)^n e^{hg}.$$

In 1971 H.M. Srivastava- J.P. Singhal [57] introduced a class of polynomials as

(1.1.43) $$G_n^{(\alpha)}(x,r,p,k) = \frac{1}{n!} x^{-\alpha-kn} e^{px^r} \theta^n (x^\alpha e^{-px^r}),$$

where $\theta = x^{k+1} D$.

In 1975 Chandel-Agarwal [27] extended Rodrigue's formula for Jacobi polynomials as,

$$(1.1.44)\qquad P_n^{(\alpha,\beta)}(x,p,r,s,c,d)$$

$$= \frac{(x^r+c)^{-\alpha}(x^s+d)^{-\beta}}{2^n\, n!}\, D^n\left\{(x^r+c)^{np+\alpha}(x^s+d)^{nq+\beta}\right\}.$$

In 1975 H.M. Srivastava-Rekha Panda [58] gave a sequence of functions as.

$$(1.1.45)\qquad S_n^{(\alpha,\beta)}(x,a,b,c,d;\,\nu,\theta;\,w(x))$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\; w(x)}\, D^n\left\{(ax+b)^{\nu n+\alpha}(cx+d)^{\theta n+\beta}\, w(x)\right\}.$$

P.N. Shrivastava [53a] gave a unified presentation of a class of polynomials as,

$$(1.1.46)\qquad P_n^{(\alpha,\beta,k)}(x,r,s,m)$$

$$= x^{-\alpha}(1-kx^r)^{-\beta/k}\, D^n\left\{x^{\alpha+mn}(1-kx^r)^{\frac{\beta}{k}+sn}\right\}.$$

Recently, P.N. Shrivastava [51] introduced a new function defined by the relation

$$(1.1.47)\qquad G_n^{(a_0;\,r,\,p)}(x;\,a_1,a_2\cdots\cdots a_n)$$

$$= x^{-(a_0+a_1+\cdots+a_n)}\, e^{px^r}\prod_{j=1}^{n}\mathfrak{D}_j\left(x^{a_0}e^{-px^r}\right),$$

where $\mathfrak{D}_j = x^{a_j}D$.

The operators xD, x^kD, $x^{k+1}D$, $\prod_{r=1}^{n} x^{a_r} D$.
etc. are frequently used by the reseachers.

Toscano [62] used the operator (xD) to define.

(1.1.48) $G_n^{(\alpha)}(x) = x^{-\alpha} e^{x} (xD)^n x^{\alpha} e^{-x}$.

In 1943 Hadwiger [32] used the operator $\left(\frac{1}{x}D\right)$ and gave an interesting result.

(1.1.49) $\left(\frac{1}{x}D\right)^n = \left(-2x^2\right)^{-n} \sum_{r=1}^{n} \frac{(2n-r-1)!}{(n-r)!\,(r-1)!} (-2x)^r D^r$.

Chak [13] defined a function as,

(1.1.50) $G_{n,k}^{(\alpha)}(x) = x^{-\alpha-nk+n} e^{x} (x^kD)^n e^{-x} x^{\alpha}$.

Chak gave a new representation to Laguerre polynomials as,

(1.1.51) $L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha-n-1} e^{x} (x^2D)^n (x^{\alpha+1} e^{-x})$.

W.A. Al-Salam [3] gave an operational representation for Laguerre and Jacobi polynomials.

(1.1.52) $\theta^n \left\{ x^{\alpha} e^{-x} \right\} = x^{\alpha+n} e^{x} \, n! \, L_n^{(\alpha)}(x)$,

(1.1.53) $\theta^n \left\{ x^{\alpha} (1-x)^{\beta+n} \right\} = x^{\alpha+n} (1-x)^{\beta} \, n! \, P_n^{(\alpha,\beta)}(1-2x)$,

where $\theta = x(1+xD)$.

R.P. Singh [54] generalized Toscano polynomials, as.

(1.1.54) $T_n^{(\alpha)}(x, r, p) = x^{-\alpha} e^{px^r} (xD)^n (x^{\alpha} e^{-px^r})$.

R.C. Chandel [25] defined another generalized Truesdell polynomials $T_n^{(\alpha,k)}(x, r, p)$ as,

$$(1.1.55)\quad T_n^{(\alpha,k)}(x,r,p) = x^{-\alpha} e^{px^r} (x^k D)^n [x^\alpha e^{-px^r}].$$

Joshi and Praja Pat [34] studied a unified representation of certain classical polynomials as

$$(1.1.56)\quad J_{n,\lambda,\mu}^{(\alpha,\beta)}(x,a,b,c,d,p,q,r)$$

$$= \frac{1}{n!\,q^n}(ax+b)^{-\alpha}(cx+d)^{-\beta} e^{px^r} D_x^n \left\{(ax+b)^{\alpha+n\lambda}(cx+d)^{\beta+n\mu} e^{-px^r}\right\},$$

where $D = \frac{d}{dx}$, a, b, c, d, p, q, r are constant and n, λ, μ are essentially non-negative integers.

P.N. Shrivastava [53] also defined a generalized function $F_n^{(r,m)}(x,a,k,p)$ by the Rodrigue's formula as

$$(1.1.57)\quad F_n^{(r,m)}(x,a,k,p) = x^{-\alpha} e^{px^r} (x^k D)^n [x^{\alpha+mn} e^{-px^r}].$$

Joshi and Prajapat [35] generalized certain classical polynomials by introducting

$$(1.1.58)\quad M_{vn}^{(\alpha)}(x,a,k)$$

$$= \frac{1}{n!}\, x^{-\alpha-nk} e^{P_v(x)} T_{a,k}^n \left(x^\alpha e^{-P_v(x)}\right),$$

where $P_v(x)$ is a polynomial in x of degree v, and K and q are constants, and

$$T_{a,k} = x^k(a+xD),\quad D \equiv \frac{d}{dx}.$$

Joshi and Prajapat [36] gave another generalized ~~the~~ class of polynomials as

$$(1.1.59)\quad M_n^{(\alpha)}(x,r,p,b,k,q)$$

$$= C(b,n)\, x^{-\alpha-nq-n} e^{px^r} T_{k,q}^n \left(x^{\alpha+bn} e^{-px^r}\right),$$

where $C(b,n)$ is a constant such that

$$C(b,n) = \frac{(-1)^{\frac{n}{2}(b-1)(b-2)}}{2^{\frac{n}{2}b(b-1)}\,(1)_{nb(2-b)}}\,.$$

and b being a non negative integer.

Recently, Patil and Thakare [45,46] have introduced the generalized function $P_n^{(\alpha)}(x,r,s,p,k,a)$ defined by Rodrigue's type formula.

(1.1.60) $P_n^{(\alpha)}(x,r,s,p,k,a) = x^{-\alpha} e^{px^r} T_{a,k}^n \left(x^{\alpha+sn} e^{-px^r}\right)$,

In 1980 Agrawal and Chaubey [2] gave a sequence of functions as

(1.1.61) $R_n^{(\alpha,\beta)}[x;a,b,c,d;p,q;\gamma,\delta;w(x)]$

$$= \frac{(ax^p+b)^{-\alpha}(cx^q+d)^{-\beta}}{K_n\,w(x)} \left\{x^k(\lambda+xD_x)\right\}^n \left\{(ax^p+b)^{\alpha+\gamma n}(cx^q+d)^{\beta+\delta n}\,w(x)\right\},$$

where $D_x \equiv \frac{d}{dx}$.

when $p=q=1$ and $w(x) = \exp(-px^r)$, (1.1.61) reduces to Thakare and Madhekar [60] in (1981).

S.K. Bhargava [8,9,11] studied a generalization of certain classes of polynomials by defining a sequence of functions.

(1.1.62) $G_n(a,k;h,g(x)) = e^{-hg(x)} T_{a,k}^n \left(e^{hg(x)}\right)$

and generalized it further in [10] as

$$G_n(a,k,p;g(x),h(x)) = e^{-pg(x)} T_{a,k}^n \left\{[h(x)]^n e^{pg(x)}\right\}.$$

where $h(x)$ and $g(x)$, are suitable functions of x and $a, k, p,$ are constants.

In the present thesis, we shall be discussing in detail, following two research papers, which deal with the class of polynomials defined by generalized Rodrigue's type formulas:

(1) Joshi, C.M. and Prajapat, M.L. : The operator $T_{k,q}$ and Characterization of a class of polynomials by the generalized Rodrigue's formula: Kyungpook Math. J. Vol.21, No. 1 June 1981.

(2) Joshi, C.M. and Prajapat, M.L. : On some properties of a class of polynomials unifying the generalized Hermite Laguerre and Bessel polynomials: The Mathematics Student Vol. 45 No.2 (1977) 74-86.

The Chapter 2 deals with paper (1) and Chapter 3 deals with paper (2).

CHAPTER-II

A CLASS OF POLYNOMIALS DEFINED BY A GENERALIZED RODRIGUE'S FORMULA

2.1 INTRODUCTION

Using the operator x^2D, where $D=\frac{d}{dx}$, in 1956 Chak [13] defined the generalized Laguerre polynomials.

(2.1.1) $$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha-n-1} e^{x} (x^2D)^n [x^{\alpha+1} e^{-x}].$$

In 1964 W.A. Al-Salam [3] gave an operational representation for Laguerre polynomials using the operator $\theta = x(1+xD)$ and showed that

(2.1.2) $$\theta^n x^{\alpha} e^{-x} = x^{\alpha+n} e^{-x} n! \, L_n^{(\alpha)}(x).$$

In 1971, Mittal [41] observed that relations (2.1.1) and (2.1.2) can, in fact, be derived from a more general operational representation. To this end he considered the operator $T_k = x(k+xD)$ being constant and showed that the polynomial set $\{T_{vn}^{(\alpha)}(x) \mid n=0,1,2------\}$ where

(2.1.3) $$T_{vn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{p_v(x)} D^n (x^{\alpha+n} e^{-p_v(x)}),$$

$p_v(x)$ is a polynomial in x of degree r, admit the relationship

(2.1.4) $$T_{vn}^{(\alpha+k-1)}(x) = \frac{1}{n!} x^{-\alpha-n} e^{p_v(x)} T_k^n (x^{\alpha} e^{-p_v(x)})$$

in terms of the operator T_k.

Motivated by these developments Joshi and Prajapat [35] considered a class of polynomials defined by

(2.1.5) $$M_{vn}^{(\alpha)}(x,k,q) = \frac{1}{n!} x^{-\alpha-nk} e^{p_v(x)} T_{k,q}^n (x^{\alpha} e^{-p_v(x)}),$$

where $P_r(x)$ is a polynomial in x of degree r and k, and q are constants, and

$$T_{k,q} = x^q(k + xD) \;, \quad D \equiv \frac{d}{dx} .$$

Subsequently, motivated by the works of Gould-Hopper [29], Singh-Srivastava [56], Chak [13] Chatterjea [24] Srivastava-Singhal, Joshi and Prajapat [35] considered a class of polynomials defined by

(2.1.6) $$M_n^{(\alpha)}(x, r, p, k, q) = \frac{1}{n!} x^{-\alpha-nq} e^{px^r} T_{k,q}^n (x^\alpha e^{-px^r}) .$$

where p, r, k and q are constants and assume integral values. The polynomials defined above happen to unify the polynomials of Laguerre, Hermite, Bessel, etc.

In particular, for k=0

$$M_n^{(\alpha)}(x, r, p, 0, q) = G_n^{(\alpha)}(x, r, p, q) .$$ the class of polynomials considered by Srivastava-Singhal.

Also,

$$M_n^{(0)}(x, 2, 1, 0, -1) = G_n^{(0)}(x, 2, 1, -1) = \frac{(-x)^n}{n!} H_n(x) .$$

and $$M_n^{(\alpha)}(x, r, p, 0, -1) = G_n^{(\alpha)}(x, r, p, -1) = \frac{(-x)^n}{n!} H_n^{(r)}(x, \alpha, p) .$$

In the present Chapter, as indicated earlier, we shall discuss the paper of Joshi-Prajapat [35]

2.2 PROPERTIES OF OPERATOR $T_{k,q}$

$T_{k,q}$ has been defined as

$$T_{k,q} = x^q(k + xD) \;, \quad D \equiv \frac{d}{dx} .$$

We list below certain operational formulas, associated with $T_{k,q}$

(2.2.1) $$T_{k,q}^n (x^{\alpha+m}) = q^n \left(\frac{\alpha+m+k}{q}\right)_n x^{\alpha+m+nq} ,$$

where $(\alpha)_n = \alpha(\alpha+1) \cdots\cdots (\alpha+n-1)$, $n \geq 1$, $(\alpha)_0 = 1$

(2.2.2) $$F(T_{k,q})[x^{\alpha} f(x)] = x^{\alpha} F[T_{k,q} + x^{q}\alpha] f(x) .$$

(2.2.3) $$F(T_{k,q})[e^{g(x)} f(x)] = e^{g(x)} F[T_{k,q} + x^{q+1} g'(x)] f(x) .$$

(2.2.4) $$T_{k,q}^{n}(x.uv) = x \sum_{m=0}^{n} \binom{n}{m} (T_{k,q}^{n-m} v)(T_{1,q}^{m} u) ,$$

where $T_{1,q} = x^{q}(1 + xD)$.

In particular,

(2.2.5) $$(T_{k,q})^{n}(u.v) = \sum_{m=0}^{n} \binom{n}{m} (T_{k,q}^{n-m} v)(T_{q}^{m} u) , \quad T_q = x^{q+1} D .$$

(2.2.6) $$e^{t T_{k,q}} [x^{\alpha} f(x)] = \frac{x^{\alpha}}{(1 - x^{q} q t)^{\left(\frac{\alpha+k}{q}\right)}} f\left[\frac{x}{(1 - x^{q} q t)^{1/q}}\right] .$$

(2.2.7) $${}_{\lambda}F_{\mu}\left[\begin{matrix}(a_{\lambda}) ; \\ (b_{\mu}) ;\end{matrix} \; t T_{k,q}\right] x^{\alpha} e^{p x^{r}}$$

$$= \sum_{j=0}^{\infty} \frac{(p)^{j} x^{\alpha+rj}}{j!} \; {}_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_{\lambda}), \left(\frac{\alpha+k+rj}{q}\right) ; \\ (b_{\mu}) ;\end{matrix} \; x^{q} q t\right]$$

where (a_{λ}) stands for the sequence of λ parameters, namely $a_1, a_2, \cdots\cdots a_{\lambda}$, with similar interpretation for (b_{μ}) .

In particular

(2.2.8) $${}_{0}F_{1}\left[- ; \frac{\alpha+k}{q} ; t T_{k,q}\right] x^{\alpha} e^{-p x^{r}}$$

$$= x^{\alpha} \sum_{m=0}^{\infty} \frac{(-p x^{r})^{m}}{m!} \; {}_{1}F_{1}\left[\frac{\alpha+k+mr}{q} ; \frac{\alpha+k}{q} ; x^{q} q t\right]$$

Proof of (2.2.1): For n=1,

$$T_{k,q}\, x^{\alpha+m} = (\alpha+m+k)\, x^{\alpha+m+q},$$

For n=2, $$T_{k,q}^{2}\, x^{\alpha+m} = q^2\left(\frac{\alpha+m+k}{q}\right)\left(\frac{\alpha+m+k}{q}+1\right) x^{\alpha+m+2q},$$

and so on by iteration, we get

$$T_{k,q}^{n}\, x^{\alpha+m} = q^n\left(\frac{\alpha+m+k}{q}\right)_n x^{\alpha+m+nq}.$$

Proof of (2.2.2): we have

$$T_{k,q}\left[x^{\alpha} f(x)\right] = x^{\alpha}\left(T_{k,q}+\alpha x^{q}\right) f(x).$$

From this by iteration, we get for any positive integer n,

$$T_{k,q}^{n}\left[x^{\alpha} f(x)\right] = x^{\alpha}\left(T_{k,q}+\alpha x^{q}\right)^{n} f(x).$$

Thus, if F(x) is a power series, we shall get

$$F(T_{k,q}) = x^{\alpha} F\left(T_{k,q}+\alpha x^{q}\right) f(x).$$

Proof of (2.2.3): We have

$$T_{k,q}\left(e^{g(x)} f(x)\right) = x^{\alpha}\left(T_{k,q}+x^{q+1} g'(x)\right) f(x).$$

On iteration, we get

$$T_{k,q}^{n}\left(e^{g(x)} f(x)\right) = x^{\alpha}\left(T_{k,q}+x^{q+1} g'(x)\right)^{n} f(x).$$

Hence, in general,

$$F(T_{k,q})\left[e^{g(x)} f(x)\right] = x^{\alpha} F\left(T_{k,q}+x^{q+1} g'(x)\right) f(x).$$

Proof of (2.2.4): For any two function u and v

$$T_{k,q}(x u v) = x^{q}(k+xD)(x.uv),$$

which on simplification gives

$$T_{k,q}(x u v) = x\left[\left(T_{k,q} u\right) v + u\left(T_{1,q} v\right)\right].$$

On iteration, we shall get

$$T_{k,q}^{n}(x u v) = x \sum_{m=0}^{n} \binom{n}{m}\left(T_{k,q}^{n-m} u\right)\left(T_{1,q}^{m} v\right).$$

Proof of (2.2.5): We have

$$T_{k,q}(u.v) = x^{q} k u v + x^{q+1}\left[(Du)v + u(Dv)\right]$$

$$\text{or} \quad T_{k,q}(u.v) = \left(T_{k,q} u\right) v + u\left(T_{q} v\right),$$

which on iteration yields

$$T_{k,q}^{n}(u.v) = \sum_{m=0}^{n} \binom{n}{m} \left(T_{k,q}^{n-m} u\right)\left(T_q^{m} v\right).$$

Proof of (2.2.6): Suppose $f(x)$ has power series expansion as $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then

$$e^{tT_{k,q}}\left[x^{\alpha} f(x)\right]$$

$$= e^{tT_{k,q}} \sum_{n=0}^{\infty} a_n x^{\alpha+n}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{t^m}{m!} \left[T_{k,q}^{m} x^{\alpha+n}\right]$$

$$= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{t^m}{m!} q^m \left(\frac{\alpha+k+n}{q}\right)_m x^{\alpha+n+mq}$$

$$= \sum_{n=0}^{\infty} a_n x^{\alpha+n} \sum_{m=0}^{\infty} \frac{(tq)^m}{m!} x^{qm} \left(\frac{\alpha+k+n}{q}\right)_m$$

$$= \sum_{n=0}^{\infty} a_n x^{\alpha+n} \left(1-qt x^q\right)^{-\left(\frac{\alpha+n+k}{q}\right)}$$

$$= \sum_{n=0}^{\infty} x^{\alpha}\left(1-qt x^q\right)^{-\left(\frac{\alpha+k}{q}\right)} a_n \left[\frac{x}{(1-qtx^q)^{1/q}}\right]$$

$$= x^{\alpha}\left(1-qtx^q\right)^{-\left(\frac{\alpha+k}{q}\right)} \sum_{n=0}^{\infty} a_n \left[\frac{x}{(1-qtx^q)^{1/q}}\right]$$

$$= x^{\alpha}\left(1-qtx^q\right)^{-\left(\frac{\alpha+k}{q}\right)} f\left[\frac{x}{(1-qtx^q)^{1/q}}\right].$$

Proof of (2.2.7): By hypergeometric expansion

$$ {}_{\lambda}F_{\mu}\left[\begin{matrix}(a_\lambda)\,; \\ (b_\mu)\,;\end{matrix}\; t\,T_{k,q}\right] x^{\alpha} e^{px^{r}} $$

$$ = \sum_{m=0}^{\infty} \frac{((a_\lambda)_m)\, t^m}{m!\,((b_\mu)_m)}\; T_{k,q}^{m}\left(x^{\alpha} e^{px^{r}}\right) $$

$$ = \sum_{m=0}^{\infty} \frac{((a_\lambda)_m)\, t^m}{m!\,((b_\mu)_m)} \sum_{j=0}^{\infty} \frac{(-p)^j}{j!}\; T_{k,q}^{m}\; x^{\alpha+rj} $$

$$ = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{((a_\lambda)_m)\, t^m}{m!\; j!\,((b_\mu)_m)}\; q^m \left(\frac{\alpha+m+k+rj}{q}\right)_m x^{\alpha+rj+mq} $$

$$ = \sum_{j=0}^{\infty} \frac{(p)^j}{j!}\, x^{\alpha+rj} \sum_{m=0}^{\infty} \frac{((a_\lambda)_m)\left(\frac{\alpha+k+m+rj}{q}\right)_m}{((b_\mu)_m)}\; \frac{(q+x^q)^m}{m!} $$

$$ = \sum_{j=0}^{\infty} \frac{(p)^j}{j!}\, x^{\alpha+rj}\; {}_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_\lambda),\; \frac{\alpha+m+k+rj}{q}\,; \\ (b_\mu)\;;\end{matrix}\; q+x^q\right]. $$

Proof of (2.2.8): In (2.2.7), replace p by $-p$ and take $\lambda=0$, $\mu=1$. we get the result.

Using the above formulas for $T_{k,q}$ we get the following operational formulas

(2.2.9) $$\prod_{j=0}^{n-1} (\delta + \alpha + k - r\rho x^{r} + jq).1$$
$$= n!\, M_n^{(\alpha)}(x, r, \rho, k, q).$$

which is equivalent to

(2.2.10) $$M_n^{(\alpha)}(x, r, \rho, k, q)$$
$$= \frac{q^n}{n!}\, e^{\rho x^{r}} \left(\frac{\delta + \alpha + k}{q}\right)_n e^{-\rho x^{r}}, \qquad \delta = xD.$$

As such we get operational relationship as

(2.2.11) $$x^{-nq}\, T_{k,q}^{n} = (\delta + k)(\delta + k + q) \cdots (\delta + k + (n-1)q).$$

<u>Proof of (2.2.9), (2.2.10) and (2.2.11)</u>: We have

$$T_{k,q} = x^q (k + xD), \qquad xD = \delta$$
$$= x^q (k + \delta).$$

Now consider

$$= T_{k,q}^{n} (x^{\alpha}. y)$$

$$= T_{k,q}^{n-1}\, T_{k,q} (x^{\alpha} y)$$

$$= T_{k,q}^{n-1}\, x^q (k + xD)(x^{\alpha} y),$$

$\theta x^{\alpha} = \alpha x^{\alpha}$, where $\theta = xD$

$$= T_{k,q}^{n-1}\, x^q (k x^{\alpha} y + \alpha x^{\alpha} y + x^{\alpha} xD.y)$$

$$= T_{k,q}^{n-1}\, x^{\alpha+q} [(\alpha + k + \delta) y]$$

$$= T_{k,q}^{n-2}\, T_{k,q} [x^{\alpha+q} (\alpha + k + \delta) y]$$

$$= T_{k,q}^{n-2}\, x^{\alpha+2q}(\alpha+k+q+\delta)(\alpha+k+\delta)\, y$$

$$= T_{k,q}^{n-3}\, x^{\alpha+3q}(\alpha+k+2q+\delta)(\alpha+k+q+\delta)(\alpha+k+\delta)\, y$$

$$\vdots$$

$$= x^{\alpha+nq}(\alpha+k+\delta)(\alpha+k+q+\delta)\cdots\cdots\cdots(\alpha+k+(n-1)q+\delta)\, y$$

$$= x^{\alpha+nq}\, q^n \left(\frac{\alpha+k+\delta}{q}\right)_n y.$$

Thus

(A) $$T_{k,q}^{n}(x^{\alpha}.y) = x^{\alpha+nq}\, q^n \left(\frac{\delta+\alpha+k}{q}\right)_n y.$$

From well known operational formula

(B) $$f(\delta)(e^{g(x)}F) = e^{g(x)}(\delta + x\, g'(x))\, F ,$$

We get

(C) $$T_{k,q}^{n}(x^{\alpha}e^{-px^{r}}f) = x^{\alpha+nq}\, q^n \left(\frac{\delta+\alpha+k}{q}\right)_n (e^{-px^{r}}.F).$$

Here if we put $f = 1$ in (C), we get from (2.2.6)

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{1}{n!}\, q^n\, e^{px^{r}} \left(\frac{\delta+\alpha+k}{q}\right)_n e^{-px^{r}},$$

which is (2.2.10).

Again using (B), we get from (C)

$$T_{k,q}^{n}(x^{\alpha}e^{-px^{r}}f) = x^{\alpha+nq}\, e^{-px^{r}}\, q^n \left(\frac{\delta+\alpha+k-prx^{r}}{q}\right)_n F.$$

From this for $f = 1$, we get

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{1}{n!} \prod_{j=0}^{n-1} (\delta+\alpha+k-prx^{r}+jq).1,$$

which proves (2.2.9).

2.3 <u>THE EXPLICIT FORM</u> - The explicit form for $M_n^{(\alpha)}(x, r, p, k, q)$ is given by

(2.3.1) $$M_n^{(\alpha)}(x, r, p, k, q)$$

$$= \frac{q^n}{n!} \sum_{m=0}^{n} \frac{(-px^{r})^m}{n!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\frac{\alpha+k+rj}{q}\right)_n .$$

To prove this, by definition, we have

$$M_n^{(\alpha)}(x,r,p,k,q)$$

$$= \frac{1}{n!} x^{-\alpha-nq} e^{px^r} T_{k,q}^n (x^\alpha e^{-px^r})$$

$$= \frac{1}{n!} x^{-\alpha-nq} e^{px^r} T_{k,q}^n \left[x^\alpha \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \right]$$

$$= \frac{1}{n!} x^{-\alpha-nq} e^{px^r} \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} T_{k,q}^n (x^{\alpha+rj}).$$

Using (2.2.1), and assuming $\frac{r}{q} = m$, a positive integer we get

$$M_n^{(\alpha)}(x,r,p,k,q)$$

$$= \frac{1}{n!} x^{-\alpha-nq} e^{px^r} \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} q^n \left(\frac{\alpha+k+rj}{q}\right)_n x^{\alpha+rj+nq}$$

$$= \frac{q^n}{n!} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+rj}{q}\right)_n$$

$$= \frac{q^n}{n!} \sum_{m=0}^{\infty} \frac{(px^r)^m}{m!} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+rj}{q}\right)_n$$

$$= \frac{q^n}{n!} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\frac{\alpha+k+rj}{q}\right)_n.$$

The inner sum is the m^{th} difference of a polynomial of degree n, hence when $m > n$, the inner sum is Zero. Thus

$$M_n^{(\alpha)}(x, r, p, k, q)$$
$$= \frac{q^n}{n!} \sum_{m=0}^{n} \frac{(-px^r)^m}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\frac{\alpha + k + rj}{q}\right)_n .$$

We observe that, for $m = \frac{r}{q}$, a positive integer

$$\left(\frac{\alpha + k}{q} + \frac{r}{q} j\right)_n = \left(\frac{\alpha + k}{q} + mj\right)_n$$
$$= \frac{\left(\frac{\alpha+k}{q}\right)_n \left(\frac{\alpha+k}{q} + n\right)_{mj}}{\left(\frac{\alpha+k}{q}\right)_{mj}} .$$

Hence

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n}{n!} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \frac{\left(\frac{\alpha+k}{q} + n\right)_{mj} \left(\frac{\alpha+k}{q}\right)_n}{\left(\frac{\alpha+k}{q}\right)_{mj}}$$

or

(2.3.2) $$M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n}{n!} e^{px^r} (a)_n \, y ,$$

where $$y = \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} \frac{(a+n)_{mj}}{(a)_{mj}} x^{rj} ,$$

and $$a = \frac{\alpha+k}{q} .$$

In terms of the difference operator $\Delta_{\alpha, r}$, we can have

(2.3.3) $$M_n^{\alpha}(x, r, p, k, q) = \frac{q^n}{n!} \sum_{m=0}^{n} \frac{(-px^r)^m}{m!} \Delta^m_{\alpha+k, r} \left(\frac{\alpha+k}{q}\right)_n ,$$

where $$\Delta_{\alpha, r} f(\alpha) = f(\alpha + r) - f(\alpha) ,$$

Equivalently we can also write

(2.3.4) $$M_n^{(\alpha)}(x, r, \rho, k, q) = \frac{q^n}{n!} \exp[-\rho x^r \Delta_{\alpha+k,r}] \left(\frac{\alpha+k}{q}\right)_n .$$

To prove (2.3.3) and (2.3.4) we use well known relation

$$\Delta^m_{\alpha,r} f(\alpha) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(\alpha + rj) .$$

Now since $\left(\frac{\alpha+k}{q}\right)_n$ is a polynomial of degree n in x, hence the m^th difference of this shall be Zero for $m > n$. Hence from (2.3.1) we get (2.3.3).

(2.3.4) is evident from (2.3.3).

From formula (2.2.5), we see that $M_n^{(\alpha)}(x, r, \rho, k, q)$ can be expressed as a polynomial of degree n in α , as

(2.3.5) $$M_n^{(\alpha)}(x, r, \rho, k, q) = \sum_{m=0}^{n} \frac{q^m}{m!} \left(\frac{\alpha}{q}\right)_m M^{(0)}_{n-m}(x, r, \rho, k, q).$$

2.4 THE DIFFERENTIAL EQUATION

For $\frac{r}{q} = m$, a positive integer the differential equations for $M_n^{(\alpha)}(x, r, \rho, k, q)$ is

(2.4.1) $$\Big[(\delta - \rho r x^r) \prod_{j=1}^{m} (\delta - \rho q r x^r + \alpha + k - r + jq - q)$$

$$+ \rho r x^r \prod_{j=1}^{m} (\delta - \rho q r x^r + \alpha + k + nq + jq - q)\Big] M_n^{(\alpha)}(x, r, \rho, k, q) = 0.$$

For $\frac{r}{q} = -m$, m is a positive integer, the differential equation is

(2.4.2) $$\Big[(\delta - \rho r x^r) \prod_{j=1}^{m} (\delta - \alpha - k - r + \rho q r x^r - jq + q)$$

$$+ \rho r x^r \prod_{j=1}^{m} (\delta - \alpha - k - nq + \rho q r x^r - jq + q)\Big] M_n^{(\alpha)}(x, r, \rho, k, q) = 0$$

<u>Proof of (2.4.1)</u>: We observe that the operator $T_{k,q}$ possesses the properties

$$x^{-q} T_{k,q} x^n = (n+k) x^n ,$$

and $x^{-q} T_{k,q} x^n = (\delta + k)$, or $(x^{-q} T_{k,q} - k) = \delta$,

In view of (2.3.2)

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n}{n!} e^{px^r} (a)_n\, y ,$$

where $a = \frac{\alpha+k}{q}$ and $y = \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} \frac{(a+n)_{mj}}{(a)_{mj}} x^{rj}$.

We have

$$\delta y = \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} \frac{(a+n)_{mj}}{(a)_{mj}} \delta x^{rj}$$

$$= \sum_{j=0}^{\infty} \frac{(-p)^j (a+n)_{mj} (rj) x^{rj}}{j! \quad (a)_{mj}} , \quad [\delta x^\alpha = \alpha x^\alpha],$$

multiply both the sides by $\left(\frac{m}{r}\delta + a - m\right)_m$

$$\delta\left(\frac{m}{r}\delta + a - m\right)_m y = \sum_{j=0}^{\infty} \frac{(-p)^j (a+n)_{mj} (rj) \left(\frac{m}{r}\delta + a - m\right)_m x^{rj}}{j!\ (a)_{mj}}$$

$$= \sum_{j=0}^{\infty} \frac{(-p)^j (a+n)_{mj} (rj) \left(\frac{m}{r} rj + a - m\right)_m x^{rj}}{j!\ (a)_{mj}}$$

$$= \sum_{j=0}^{\infty} \frac{(-p)^j (a+n)_{mj} (rj) (mj + a - m)_m x^{rj}}{j!\ (a)_{mj}} .$$

Since

$$(a)_{mj} = (a)_{m(j-1)} (a + mj - 1) ,\ \text{We have}$$

$$\delta\left(\frac{m}{r}\delta + a - m\right)_m y = r \sum_{j=0}^{\infty} \frac{(-p)^j (a+n)_{mj} (a + mj - m)_m x^{rj}}{(j-1)!\ (a)_{mj}}$$

$$= r \sum_{j=0}^{\infty} \frac{(-p)^{j+1} (a+n)_{m(j+1)} [a + m(j+1) - m]_m x^{r(j+1)}}{j!\ (a)_{mj+m}}$$

$$= r \sum_{j=0}^{\infty} \frac{(-p)^{j+1} (a+n)_{mj+m} \, (a+mj)_m}{j! \, (a)_{mj} \, (a+mj)_m} x^{rj+r}$$

$$= r \sum_{j=0}^{\infty} \frac{(-p)^{j+1} (a+n)_{mj+m}}{j! \, (a)_{mj}} x^{rj+r}$$

$$= r \sum_{j=0}^{\infty} \frac{(-p)^{j} (-px^r) \, (a+n)_{mj+m}}{j! \, (a)_{mj}} x^{rj}$$

$$= -prx^r \sum_{j=0}^{\infty} \frac{(-p)^{j} (a+n)_{mj} \, (a+n+mj)_m}{j! \, (a)_{mj}} x^{rj}$$

$$= -px^r r \sum_{j=0}^{\infty} \frac{(-p)^{j} (a+n)_{mj} \, (\frac{m}{r}\delta + a + n)_m}{j! \, (a)_{mj}} x^{rj}$$

$$= -prx^r \left(\frac{m}{r}\delta + a + n\right)_m \sum \frac{(-p)^{j} (a+n)_{mj}}{j! \, (a)_{mj}} x^{rj}$$

$$= -prx^r \left(\frac{m}{r}\delta + a + n\right)_m y .$$

Therefore,

$$\delta\left(\frac{m}{r}+a-m\right)_m y = -p r x^r \left(\frac{m}{r}\delta + a + n\right)_m y.$$

Now since

$$y = \frac{n!}{(a)_n q^n} e^{-px^r} M_n^{(\alpha)}(x, r, p, k, q),$$

We have

$$\frac{n!}{(a)_n q^n}\delta\left(\frac{m}{r}\delta+a-m\right)_m\left[e^{-px^r} M_n^{(\alpha)}(x,r,p,k,q)\right]$$
$$= \frac{n!}{(a)_n q^n}(-prx^r)\left(\frac{m}{r}\delta+a+n\right)_m\left[e^{-px^r} M_n^{(\alpha)}(x,r,p,k,q)\right];$$

or

$$(\delta - prx^r)\left(\frac{m}{r}\delta+a-m-prx^r\right)_m M_n^{(\alpha)}(x,r,p,k,q)$$
$$= -prx^r\left(\frac{m}{r}\delta+a+n-prx^r\right)_m M_n^{(\alpha)}(x,r,p,k,q).$$

This can also be equivalently expressed as

$$\text{(2.4.3)}\quad \Big[\left(x^{-q}T_{k,q}-k-prx^r\right)\left(\frac{m}{r}(x^{-q}T_{k,q}-k)+a-m-prx^r\right)_m$$
$$+ prx^r\left(\frac{m}{r}(x^{-q}T_{k,q}-k)+a+n-prx^r\right)_m\Big]M_n^{(\alpha)}(x,r,p,k,q)=0.$$

Replacing a by $\frac{\alpha+k}{q}$, and m by $\frac{r}{q}$, we get

$$\Big[(\delta - prx^r)\prod_{j=1}^{m}(\delta+\alpha+k-r-pqrx^r+jq-q)$$
$$+ prx^r\prod_{j=1}^{m}(\delta+\alpha+k-pqrx^r+nq+jq-q)\Big]M_n^{(\alpha)}(x,r,p,k,q)=0,$$

which is (2.4.1).

<u>Proof of (2.4.2):</u>

If $\frac{r}{q} = -m$, where m is a positive integer, we easily get

$$\left(\frac{m}{r}\delta - prx^r + a - m\right)_m = \frac{(-1)^m}{q^m}\prod_{j=1}^{m}\left(\delta - \alpha - k - r + pqrx^r - jq + q\right)$$

Similarly

$$\left(\frac{m}{r}\delta - prx^r + a + n\right)_m = \frac{(-1)^m}{q^m}\prod_{j=1}^{m}\left(\delta - \alpha - k - nq + pqrx^r - jq + q\right).$$

Hence the differential equation assumes the form

$$\Big[\left(\delta - prx^r\right)\frac{(-1)^m}{q^m}\prod_{j=1}^{m}\left(\delta - \alpha - k + pqrx^r - jq + q\right)$$
$$+ prx^r\frac{(-1)^m}{q^m}\prod_{j=1}^{m}\left(\delta - \alpha - k - nq + pqrx^r + jq - q\right)\Big]M_n^{(\alpha)}(x,r,p,k,q) = 0,$$

or

$$\Big[\left(\delta - prx^r\right)\prod_{j=1}^{m}\left(\delta - \alpha - k + pqrx^r - jq + q\right)$$
$$+ prx^r\prod_{j=1}^{m}\left(\delta - \alpha - k - nq + pqrx^r - jq + q\right)\Big]M_n^{(\alpha)}(x,r,p,k,q) = 0,$$

which is (2.4.2).

2.5 GENERATING FUNCTIONS - For $M_n^{(\alpha)}(x,r,p,k,q)$ we have the following generating relations.

(2.5.1) $$\sum_{n=0}^{\infty}\frac{[(a_\lambda)]_n}{[(b_\mu)]_n}M_n^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$= e^{px^r}\sum_{n=0}^{\infty}\frac{(-px^r)^n}{n!}\;{}_{\lambda+1}F_\mu\left[\begin{matrix}(a_\lambda), & \frac{\alpha+k+nr}{q}; \\ (b_\mu) & ;\end{matrix}\; qt\right].$$

(2.5.2) $$\sum_{n=0}^{\infty}M_n^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)}\exp\left[px^r\left\{1-(1-qt)^{-r/q}\right\}\right].$$

(2.5.3) $$\sum_{n=0}^{\infty}M_n^{(\alpha-nq)}(x,r,p,k,q)\,t^n$$
$$= (1+qt)^{\left(\frac{\alpha+k-q}{q}\right)}\exp\left[px^r\left\{1-(1+qt)^{r/q}\right\}\right].$$

$$\sum_{n=0}^{\infty} \frac{[(a_\lambda)]_n}{[(b_\mu)]_n} M_n^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{[(a_\lambda)]_n}{[(b_\mu)]_n} \frac{1}{n!} x^{-\alpha-nq} e^{px^r} T_{k,q}^{n} (x^\alpha e^{-px^r})\, t^n$$

$$= x^{-\alpha} e^{px^r} \sum_{n=0}^{\infty} \frac{[(a_\lambda)]_n}{[(b_\mu)]_n} \frac{x^{-nq}}{n!} T_{k,q}^{n} (x^\alpha e^{-px^r})\, t^n$$

$$= x^{-\alpha} e^{px^r} \sum_{n=0}^{\infty} \frac{[(a_\lambda)]_n}{[(b_\mu)]_n} \frac{(t x^{-q})^n}{n!} T_{k,q}^{n} (x^\alpha e^{-px^r})$$

$$= x^{-\alpha} e^{px^r} \sum_{n=0}^{\infty} \frac{[(a_\lambda)]_n}{[(b_\mu)]_n} \frac{(t x^{-q} T_{k,q})^n}{n!} (x^\alpha e^{-px^r})$$

$$= x^{-\alpha} e^{px^r} {}_\lambda F_\mu \left[\begin{matrix} (a_\lambda); \\ (b_\mu); \end{matrix} \; t x^{-q} T_{k,q} \right] x^\alpha e^{-px^r}.$$

Now by (2.2.7), we have

$${}_\lambda F_\mu \left[\begin{matrix} (a_\lambda); \\ (b_\mu); \end{matrix} \; t T_{k,q} \right] x^\alpha e^{px^r}$$

$$= \sum_{j=0}^{\infty} \frac{(p)^j}{j!} x^{\alpha+rj} {}_{\lambda+1}F_\mu \left[\begin{matrix} (a_\lambda), \dfrac{\alpha+k+rj}{q}; \\ (b_\mu) \; ; \end{matrix} \; x^q q t \right]$$

$$= x^{-\alpha} e^{px^r} \sum_{n=0}^{\infty} \frac{(-p)^n}{n!} x^{\alpha+rn} {}_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_\lambda), & \frac{\alpha+k+rn}{q}; \\ (b_\mu) & ; \end{matrix}\; qt\right]$$

$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} {}_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_\lambda), & \frac{\alpha+k+nr}{q}; \\ (b_\mu) & ; \end{matrix}\; tq\right].$$

Hence we get

$$\sum_{n=0}^{\infty} \frac{[(a_\lambda)]_n}{[(b_\mu)]_n} M_n^{(\alpha)}(x,r,p,k,q)\, t^n$$
$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} {}_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_\lambda), & \frac{\alpha+k+rn}{q}; \\ (b_\mu) & ; \end{matrix}\; qt\right],$$

which proves (2.5.1).

Now in particular for $\lambda=\mu$. $a_i = b_j$. $j=1,2 \cdots \lambda$ (or μ) then (2.5.1) reduces to

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,q)\, t^n$$
$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} {}_1F_0\left[\begin{matrix}\frac{\alpha+k+rn}{q}; \\ - \;; \end{matrix}\; qt\right]$$
$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} \sum_{j=0}^{\infty} \left(\frac{\alpha+k+rn}{q}\right)_j \frac{(qt)^j}{j!}$$
$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} (1-qt)^{-\left(\frac{\alpha+k+rn}{q}\right)}$$
$$= e^{px^r} (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \sum_{n=0}^{\infty} \frac{\left[-px^r(1-qt)^{-r/q}\right]^n}{n!}$$
$$= e^{px^r} (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[-px^r(1-qt)^{-r/q}\right]$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right],$$

which proves (2.5.2).

<u>Proof of (2.5.3)</u>

By explicit formula,

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, t^n = \sum_{n=0}^{\infty} \frac{q^n}{n!} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+rj}{q}\right)_n t^n ,$$

Replacing α by $(\alpha - nq)$, we get

$$\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x, r, p, k, q)\, t^n = \sum_{n=0}^{\infty} \frac{q^n}{n!} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha-nq+rj+k}{q}\right)_n t^n .$$

Now

$$\left(\frac{\alpha+k+rj-nq}{q}\right)_n = \left(\frac{\alpha+k+rj}{q} - n\right)_n = (-1)^n \left(1 - \frac{\alpha+k+rj}{q}\right)_n .$$

Hence

$$\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x, r, p, k, q)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{(qt)^n}{n!} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} (-1)^n \left(1 - \frac{\alpha+k+rj}{q}\right)_n$$

$$= e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \sum_{n=0}^{\infty} \left(1 - \frac{\alpha+k+rj}{q}\right)_n \frac{(-qt)^n}{n!}$$

$$= e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} (1+qt)^{-\left(1 - \frac{\alpha+k+rj}{q}\right)}$$

$$= e^{px^r}(1+qt)^{\left(\frac{\alpha+k-q}{q}\right)} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left[(1+qt)^{r/q}\right]^j$$

$$= e^{px^r}(1+qt)^{\left(\frac{\alpha+k-q}{q}\right)} \exp\left[-px^r(1+qt)^{r/q}\right]$$

$$= (1+qt)^{\left(\frac{\alpha+k-q}{q}\right)} \exp\left[px^r\left\{1-(1+qt)^{r/q}\right\}\right],$$

which proves (2.5.3).

<u>Proof of (2.5.4)</u>

Replacing α by $(\alpha-nq)$, we have

$$M_n^{(\alpha)}(x,r,p,k,q) = \frac{1}{n!} x^{-\alpha} e^{px^r} T_{k,q}^n \left(x^{\alpha-nq} e^{-px^r}\right).$$

Multiplying (2.5.3) by $x^\alpha e^{-px^r}$ and operating by $T_{k,q}^m$ we get

$$\sum_{n=0}^{\infty} T_{k,q}^m x^\alpha e^{-px^r} M_n^{(\alpha-nq)}(x,r,p,k,q)$$
$$= T_{k,q}^m x^\alpha e^{-px^r}(1-qt)^{\left(\frac{\alpha+k-q}{q}\right)} \exp\left[px^r\left\{1-(1-qt)^{r/q}\right\}\right].$$

Now L.H.S.

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} T_{k,q}^m x^\alpha e^{-px^r} x^{-\alpha} e^{px^r} T_{k,q}^n \left(x^{\alpha-nq} e^{-px^r}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} T_{k,q}^{m+n} \left(x^{\alpha-nq} e^{-px^r}\right) t^n$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} (m+n)!\, x^{\alpha+mq} e^{-px^r} M_{m+n}^{(\alpha-nq)}(x,r,p,k,q)$$

$$= m!\, x^{\alpha+mq} e^{-px^r} \sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha-nq)}(x,r,p,k,q)\, t^n.$$

Also R.H.S.

$$= T_{k,q}^m \left[x^\alpha (1+qt)^{\left(\frac{\alpha+k-q}{q}\right)} \exp\left\{-px^r(1+qt)^{r/q}\right\}\right]$$

$$= (1+qt)^{\left(\frac{k-q}{q}\right)} T_{k,q}^m \left[x^\alpha (1+qt)^{\alpha/q} \exp\left\{-px^r(1+qt)^{r/q}\right\}\right]$$

$$= (1+qt)^{\left(\frac{k-q}{q}\right)} T_{k,q}^{m} \left[\{x(1+qt)^{1/q}\}^{\alpha} \exp\{-p[x(1+qt)^{1/q}]^{r}\} \right]$$

$$= (1+qt)^{\left(\frac{k-q-mq}{q}\right)} m!\, y^{\alpha+mq} e^{-py^{r}} M_{m}^{(\alpha)}(y, r, p, k, q),$$

Where $\quad y = \dfrac{x}{(1+qt)^{-1/q}}$,

$$= m!\,(1+qt)^{\left(\frac{\alpha+k-q}{q}\right)} x^{\alpha+mq} \exp[-px^{r}(1+qt)^{r/q}]\, M_{m}^{(\alpha)}\left(\frac{x}{(1+qt)^{-1/q}}, r, p, k, q\right).$$

Thus equating the two sides, we get (2.5.4).

<u>Proof of (2.5.5)</u>

Consider,

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha)}(x, r, p, k, q)\, t^{n}$$

$$= \sum_{n=0}^{\infty} \binom{m+n}{n} \frac{t^{n}}{(m+n)!} x^{-\alpha-(m+n)q} e^{px^{r}} T_{k,q}^{m+n}(x^{\alpha} e^{-px^{r}})$$

$$= \sum_{n=0}^{\infty} \frac{t^{n}}{n!} x^{-\alpha-(m+n)q} e^{px^{r}} T_{k,q}^{n} \left\{ \frac{1}{m!} T_{k,q}^{m}(x^{\alpha} e^{-px^{r}}) \right\}$$

$$= \sum_{n=0}^{\infty} \frac{t^{n}}{n!} x^{-\alpha-(m+n)q} e^{px^{r}} T_{k,q}^{n} \left\{ x^{\alpha+mq} e^{-px^{r}} M_{m}^{(\alpha)}(x, r, p, k, q) \right\}$$

$$= x^{-\alpha-mq} e^{px^{r}} \sum_{n=0}^{\infty} \frac{t^{n}}{n!} x^{-nq} T_{k,q}^{n} \left\{ x^{\alpha+mq} e^{-px^{r}} M_{m}^{(\alpha)}(x, r, p, k, q) \right\}$$

$$= x^{-\alpha-mq} e^{px^{r}} \sum_{n=0}^{\infty} \frac{(tx^{-q} T_{k,q})^{n}}{n!} \left\{ x^{\alpha+mq} e^{-px^{r}} M_{m}^{(\alpha)}(x, r, p, k, q) \right\}$$

$$= x^{-\alpha-mq} e^{px^{r}} \exp[tx^{-q} T_{k,q}] \left\{ x^{\alpha+mq} e^{-px^{r}} M_{m}^{(\alpha)}(x, r, p, k, q) \right\}$$

$$= x^{-\alpha-mq} e^{px^{r}} \exp[y T_{k,q}] \left\{ x^{\alpha+mq} e^{-px^{r}} M_{m}^{(\alpha)}(x, r, p, k, q) \right\}, \quad (tx^{-q} = y)$$

$$= x^{-\alpha-mq} e^{px^r} \frac{x^{\alpha+mq}}{(1-x^q qy)^{\left(\frac{\alpha+k+mq}{q}\right)}} \exp\left[-px^r(1-x^q qy)^{-r/q}\right].$$

$$\cdot M_m^{(\alpha)}\left(\frac{x}{(1-x^q qy)^{1/q}}, r, p, k, q\right)$$

$$= e^{px^r}(1-qt)^{-\left(\frac{\alpha+k}{q}\right)-m} \exp\left[-px^r(1-qt)^{-r/q}\right] M_m^{(\alpha)}\left(\frac{x}{(1-qt)^{1/q}}, r, p, k, q\right)$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)-m} \exp\left[px^r\{1-(1-qt)^{-r/q}\}\right] M_m^{(\alpha)}\left(\frac{x}{(1-qt)^{1/q}}, r, p, k, q\right).$$

Hence (2.5.5) is proved

<u>Proof of (2.5.6) and (2.5.7)</u>

$$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q)$$

$$= \sum_{n=0}^{\infty} \frac{(x^q t)^n}{n!} \frac{x^{-\alpha-nq}}{\left(\frac{\alpha+k}{q}\right)_n} e^{px^r} T_{k,q}^n (x^\alpha e^{-px^r})$$

$$= x^{-\alpha} e^{px^r} \sum_{n=0}^{\infty} \frac{1}{\left(\frac{\alpha+k}{q}\right)_n} \frac{(t T_{k,q})^n}{n!} (x^\alpha e^{-px^r})$$

$$= x^{-\alpha} e^{px^r} {}_0F_1\left[-; \frac{\alpha+k}{q}; t T_{k,q}\right] x^\alpha e^{-px^r},$$

This proves (2.5.6).

Now using (2.2.8), we get

$$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q) = e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} {}_1F_1\left[\frac{\alpha+k+mr}{q}; \frac{\alpha+k}{q}; x^q qt\right].$$

Using Kummer's formula,

$${}_1F_1[a;b;z] = e^z \, {}_1F_1[(b-a);b;-z],$$ we get ,

$$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x,r,p,k,q)$$
$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} e^{x^q q t} \, {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; -x^q q t\right],$$

or

$$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x,r,p,k,q)$$
$$= e^{px^r + x^q q t} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; -x^q q t\right].$$

Replacing t by $x^{-q}t$, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x,r,p,k,q) = e^{px^r + qt} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; -qt\right],$$

This proves (2.5.7).

<u>Proof of (2.5.8)</u>: From (2.2.8), if $\frac{r}{q} = s$, a positive integer, we notice that

$${}_0F_1\left[-; \frac{\alpha+k}{q}; t T_{k,q}\right] x^\alpha e^{-px^r}$$
$$= x^\alpha \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_1F_1\left[\frac{\alpha+k+mr}{q}; \frac{\alpha+k}{q}; x^q q t\right]$$
$$= x^\alpha \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha+k+mr}{q}\right)_n}{\left(\frac{\alpha+k}{q}\right)_n} \frac{(x^q q t)^n}{n!}$$
$$= x^\alpha \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha+k+nq}{q}\right)_{sm}}{\left(\frac{\alpha+k}{q}\right)_{sm}} \frac{(x^q q t)^n}{n!},$$

as $$\left(\frac{\alpha+k+mr}{q}\right)_n = \left(\frac{\alpha+k}{q}\right)_n \left(\frac{\alpha+k}{q}+n\right)_{m\frac{r}{q}} \Big/ \left(\frac{\alpha+k}{q}\right)_{m\frac{r}{q}}.$$

Since $$(\alpha)_{nk} = k^{nk} \prod_{j=1}^{k} \left(\frac{\alpha+j-1}{k}\right)_n,$$

$${}_0F_1\left[-; \frac{\alpha+k}{q}; t T_{k,q}\right] x^\alpha e^{-px^r}$$
$$= x^\alpha \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{s^{sm} \prod_{j=1}^{s} \left(\frac{\alpha+k+nq+j-1}{sq}\right)_m}{s^{sm} \prod_{j=1}^{s} \left(\frac{\alpha+k+j-1}{q}\right)_m} \frac{(x^q q t)^n}{n!}$$

$$= x^{\alpha} \sum_{n=0}^{\infty} \frac{(x^q q t)^n}{n!} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{s} \left(\frac{\alpha+k+nq+j-1}{sq}\right)_m}{\prod_{j=1}^{s} \left(\frac{\alpha+k+j-1}{sq}\right)_m} \frac{(-px^r)^m}{m!}$$

$$= x^{\alpha} \sum_{n=0}^{\infty} \frac{(x^q q t)^n}{n!} {}_sF_s \left[\begin{matrix} \Delta(s, \frac{\alpha+k+nq}{q}) ; \\ \Delta(s, \frac{\alpha+k}{q}) ; \end{matrix} -px^r \right],$$

Where $\Delta(s,\alpha)$ stands for S parameter $\frac{\alpha}{s}, \frac{\alpha+1}{s} \; ...... \; \frac{\alpha+s-1}{s}$.

Hence from (2.5.6)

(2.5.9) $$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q)$$

$$= x^{\alpha} \sum_{n=0}^{\infty} \frac{(x^q q t)^n}{n!} {}_sF_s \left[\begin{matrix} \Delta(s, \frac{\alpha+k+nq}{q}) ; \\ \Delta(s, \frac{\alpha+k}{q}) ; \end{matrix} -px^r \right].$$

Now equating the coefficient of $(x^q t)^n$ on both the sides we get

(2.5.10) $$M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n \left(\frac{\alpha+k}{q}\right)_n}{n!} e^{px^r} {}_sF_s \left[\begin{matrix} \Delta(s, \frac{\alpha+k+nq}{q}) ; \\ \Delta(s, \frac{\alpha+k}{q}) ; \end{matrix} -px^r \right]$$

which gives another hypergeometric form for $M_n^{(\alpha)}(x, r, p, k, q)$.

Next, if we put $\lambda = \mu = 1$ and $a_1 = c/q$ or $b_1 = \frac{\alpha+k}{q}$ then (2.5.1) reduces to,

$$\sum_{n=0}^{\infty} \frac{(c/q)_n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} {}_2F_1 \left[\begin{matrix} c/q, \frac{\alpha+k+nr}{q} ; \\ \frac{\alpha+k}{q} ; \end{matrix} qt \right].$$

From the hypergeometric transformation

$$F\left[\begin{matrix} a, b; \\ c ; \end{matrix} z \right] = (1-z)^{-a} F\left[\begin{matrix} a, c-b; \\ c ; \end{matrix} -\frac{z}{(1-z)} \right], \quad |z| < 1, \; \left|\frac{z}{1-z}\right| < 1;$$

We get

L.H.S. $$= e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} (1-qt)^{-\frac{c}{q}} {}_2F_1 \left[\begin{matrix} \frac{c}{q}, -\frac{nr}{q} ; \\ \frac{\alpha+k}{q} ; \end{matrix} \frac{qt}{(qt-1)} \right].$$

Hence

$$\sum_{n=0}^{\infty} \frac{(c/q)_n}{\left(\frac{\alpha+k}{q}\right)_n} M^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$= (1-qt)^{-\frac{c}{q}} e^{px^r} \sum_{n=0}^{\infty} \frac{(-px^r)^n}{n!} {}_2F_1 \left[\begin{matrix} -\frac{nr}{q}, \frac{c}{q} ; \\ \frac{\alpha+k}{q} ; \end{matrix} \frac{qt}{(qt-1)} \right],$$

This proves (2.5.8).

2.6 SOME APPLICATIONS OF GENERATING FUNCTIONS

Following recurrence and other relations are satisfied by

(2.6.1) $$M_n^{(\alpha-nq)}(x,r,\rho,\kappa,q) + q\,M_{n-1}^{(\alpha-(n-1)q)}(x,r,\rho,\kappa,q) = M_n^{(\alpha)}(x,r,\rho,\kappa,q).$$

(2.6.2) $$D M_n^{(\alpha)}(x,r,\rho,\kappa,q) = r\rho x^{r-1}\left[M_n^{(\alpha)}(x,r,\rho,\kappa,q) - M_n^{(\alpha)}(x,r,\rho,\kappa,q)\right].$$

(2.6.3) $$(\alpha+\kappa-q)\,M_n^{(\alpha)}(x,r,\rho,\kappa,q) = (n+1)\,M_{n+1}^{(\alpha-q)}(x,r,\rho,\kappa,q) + \rho r x^{r} M_n^{(\alpha+r)}(x,r,\rho,\kappa,q).$$

(2.6.4) $$M_n^{(\alpha)}(x,r,m\rho,\kappa,q) = M_n^{(\alpha)}(m^{1/r}x,r,\rho,\kappa,q).$$

(2.6.5) $$M_n^{\left(\sum_{s=1}^{m}\alpha_s\right)}\left[x,r,\sum_{s=1}^{m}\rho_s,m\kappa,q\right] = \sum_{i_1+\cdots i_m=n}\prod_{j=1}^{m} M_{i_j}^{(\alpha_j)}(x,r,\rho_j,\kappa,q).$$

(2.6.6) $$M_n^{\left(\sum_{s=1}^{m}\alpha_s\right)}\left[\left(\sum_{s=1}^{m}x_s\right)^{1/r},r,\rho,m\kappa,q\right] = \sum_{i_1+\cdots i_m=n}\prod_{j=1}^{m} M_{i_j}^{(\alpha_j)}\left[(x_j)^{1/r},r,\rho,\kappa,q\right].$$

(2.6.7) $$M_n^{(\alpha)}(x,r,\rho,\kappa,q) = \sum_{m=0}^{\infty}\frac{\left(\frac{\alpha-\beta}{q}\right)_m}{m!}\,q^m M_{n-m}^{(\beta)}(x,r,\rho,\kappa,q).$$

(2.6.8) $$T_{\kappa,q}^{n}\left[e^{-\rho x^{r}}x^{\alpha+nq}M_n^{(\alpha)}(x,r,\rho,\kappa,q)\right] = \frac{(m+n)!}{n!}\,x^{\alpha+(m+n)q}e^{-\rho x^{r}}M_{m+n}^{(\alpha)}(x,r,\rho,\kappa,q).$$

(2.6.9) $$\left[T_{\kappa,q}+(\alpha+nq)x^{q}-\rho r x^{r+q}\right]^{m}M_n^{(\alpha)}(x,r,\rho,\kappa,q) = \frac{(m+n)!}{n!}\,x^{mq}M_{m+n}^{(\alpha)}(x,r,\rho,\kappa,q).$$

(2.6.10) $$(n+1)\,M_{n+1}^{(\alpha)}(x,r,\rho,\kappa,q) = (\alpha+\kappa+nq)\,M_n^{(\alpha)}(x,r,\rho,\kappa,q) - \rho r x^{r}M_n^{(\alpha+r)}(x,r,\rho,\kappa,q).$$

To prove the above relations we shall appeal to the generating relations, discussed in the previous section.

Proof: Replacing q by $-q$ in (2.5.2), we get

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,-q)\,t^n = (1+qt)^{\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\left\{1-(1+qt)^{r/q}\right\}\right],$$

and from (2.5.3), we have

$$\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x,r,p,k,q)\,t^n = (1+qt)^{\left(\frac{\alpha+k}{q}\right)-1} \exp\left[px^r\left\{1-(1+qt)^{r/q}\right\}\right]$$

$$(1+qt)\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x,r,p,k,q)\,t^n = (1+qt)^{\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\left\{1-(1+qt)^{r/q}\right\}\right]$$

$$(1+qt)\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x,r,p,k,q)\,t^n = \sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,-q)\,t^n$$

$$\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x,r,p,k,q)\,t^n + q\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x,r,p,k,q)\,t^{n+1}$$
$$= \sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,-q)\,t^n.$$

Now equating the coefficients of t^n on both sides, we get

$$M_n^{(\alpha-nq)}(x,r,p,k,q) + q\,M_{n-1}^{(\alpha-(n-1)q)}(x,r,p,k,q) = M_n^{(\alpha)}(x,r,p,k,-q),$$

which proves (2.6.1).

(From (2.5.2)

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,q)\,t^n = (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\left\{1-(1-qt)^{-r/q}\right\}\right].$$

Differentiating with respect to x on both sides, we get

$$\sum_{n=0}^{\infty} D\,M_n^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)}\left[rpx^{r-1}\left\{1-(1-qt)^{-r/q}\right\}\right].$$
$$\cdot\exp\left[px^r\left\{1-(1-qt)^{-r/q}\right\}\right], \quad \left(D\equiv\frac{d}{dx}\right)$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \left[\left(r p x^{r-1} - r p x^{r-1} (1-qt)^{-r/q} \right) \right] . \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right]$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \left(r p x^{r-1} \right) \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right] . - r p x^{r-1} (1-qt)^{-\left(\frac{\alpha+k+r}{q}\right)} \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right]$$

$$= r p x^{r-1} \sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, t^n - r p x^{r-1} \sum_{n=0}^{\infty} M_n^{(\alpha+r)}(x, r, p, k, q)\, t^n ,$$

by use of (2.5.2) and adjustment of parameters.

Now equating the coefficients of t^n on both sides, we get

$$D M_n^{(\alpha)}(x, r, p, k, q) = r p x^{r-1} \left[M_n^{(\alpha)}(x, r, p, k, q) - M_n^{(\alpha+k)}(x, r, p, k, q) \right],$$

which proves (2.6.2).

Again differentiating (2.5.2) w.r.t.t, we get

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, n\, t^{n-1}$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \left[- p x^r \left(-\frac{r}{q} \right) (1-qt)^{-\frac{r}{q}-1} (-q) \right] . \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right] + \left(-\frac{\alpha+k}{q} \right) (-q) (1-qt)^{-\left(\frac{\alpha+k}{q}\right)-1} . \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right]$$

$$= - p r x^r (1-qt)^{-\left(\frac{\alpha+k+r+q}{q}\right)} \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right] + (\alpha+k)(1-qt)^{-\left(\frac{\alpha+k+q}{q}\right)} \exp\left[p x^r \left\{ 1 - (1-qt)^{-r/q} \right\} \right]$$

$$= - p r x^r \sum_{n=0}^{\infty} M_n^{(\alpha+r+q)}(x, r, p, k, q)\, t^n + (\alpha+k) \sum_{n=0}^{\infty} M_n^{(\alpha+q)}(x, r, p, k, q)\, t^n .$$

Equating the coefficients of t^n on both sides, we get

$$(n+1)\, M_{n+1}^{(\alpha)}(x,r,p,k,q)$$
$$= -prx^r M_n^{(\alpha+r+q)}(x,r,p,k,q) + (\alpha+k)\, M_n^{(\alpha+q)}(x,r,p,k,q).$$

Replacing α by $(\alpha-q)$, it changes to

$$(\alpha+k-q)\, M_n^{(\alpha)}(x,r,p,k,q)$$
$$= (n+1)\, M_{n+1}^{(\alpha-q)}(x,r,p,k,q) + prx^r M_n^{(\alpha+r)}(x,r,p,k,q),$$

which proves (2.6.3).

Expression (2.6.1), (2.6.2) and (2.6.3) are recursion formulas.

From the generating relation (2.5.2), we get

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,mp,k,q)\, t^n$$
$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[(mp).\, x^r\left\{1-(1-qt)^{-r/q}\right\}\right]$$
$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[p\,(m^{1/r}x)^r\left\{1-(1-qt)^{-r/q}\right\}\right]$$
$$= \sum_{n=0}^{\infty} M_n^{(\alpha)}(m^{1/r}x,r,p,k,q)\, t^n,$$

which on equating the coefficients of t^n on both sides gives

$$M_n^{(\alpha)}(x,r,mp,k,q) = M_n^{(\alpha)}(m^{1/r}x,r,p,k,q),$$

This proves (2.6.4).

From (2.5.2)

$$\sum_{n=0}^{\infty} M_n^{\left(\sum_{i=1}^{m}\alpha_i\right)}\left(x,r,\sum_{i=1}^{m}p_i,mk,q\right) t^n$$
$$= (1-qt)^{-\left(\frac{\Sigma\alpha_i+mk}{q}\right)} \exp\left[\left(\Sigma p_i\right)x^r\left\{1-(1-qt)^{-r/q}\right\}\right]$$
$$= \prod_{i=1}^{m} (1-qt)^{-\left(\frac{\alpha_i+m}{q}\right)} \exp\left[p_i x^r\left\{1-(1-qt)^{-r/q}\right\}\right]$$

Thus equating the coefficients of t^n on both sides, we get

$$M_n^{(\alpha)}(x,r,p,k,q) = \sum_{m=0}^{n} \frac{\left(\frac{\alpha-\beta}{q}\right)_m}{m!} q^m M_{n-m}^{(\beta)}(x,r,p,k,q).$$

This proves (2.6.7).

Consider, $T_{k,q}^{n}\left[e^{-px^r} x^{\alpha+nq} M_n^{(\alpha)}(x,r,p,k,q)\right.$

$$= \frac{1}{n!} T_{k,q}^{m+n}\left(x^{\alpha} e^{-px^r}\right)$$

$$= \frac{(m+n)!}{n!} x^{\alpha+(m+n)q} e^{-px^r} M_{m+n}^{(\alpha)}(x,r,p,k,q),$$

which proves (2.6.8).

Again from (2.2.2), Put $f(x) = e^{-px^r} f(x)$,

$$F(T_{k,q})\left[x^{\alpha} e^{-px^r} f(x)\right] = x^{\alpha} F\left[T_{k,q} + x^q \alpha\right] e^{-px^r} f(x),$$

Now using (2.2.3), we get

$$F(T_{k,q})\left[x^{\alpha} e^{-px^r} f(x)\right]$$

$$= x^{\alpha} e^{-px^r} F\left[T_{k,q} + x^q \alpha + x^{q+1}(-prx^{r-1})\right] f(x)$$

$$= x^{\alpha} e^{-px^r} F\left[T_{k,q} + x^q \alpha - prx^{r+q}\right] f(x),$$

or

$$\text{(D)} \qquad F\left[T_{k,q} + x^q \alpha - prx^{r+q}\right] f(x) = x^{-\alpha} e^{px^r} F(T_{k,q})\left[x^{\alpha} e^{-px^r} f(x)\right].$$

Consider,

$$\left[T_{k,q} + (\alpha+nq)x^q - prx^{r+q}\right]^m M_n^{(\alpha)}(x,r,p,k,q)$$

Now using (D), we get

$$= x^{-\alpha-nq} e^{px^r} T_{k,q}^{m}\left[x^{\alpha+nq} e^{-px^r} M_n^{(\alpha)}(x,r,p,k,q)\right]$$

$$= x^{-\alpha-nq} e^{px^r} \frac{1}{n!} T_{k,q}^{m+n}\left(x^{\alpha} e^{-px^r}\right)$$

$$= x^{-\alpha-nq} e^{px^r} \frac{(m+n)!}{n!} x^{\alpha+(m+n)q} e^{-px^r} M_{m+n}^{(\alpha)}(x, r, p, k, q)$$

$$= \frac{(m+n)!}{n!} x^{mq} M_{m+n}^{(\alpha)}(x, r, p, k, q),$$

which proves (2.6.9).

Letting m=1, (2.6.9) reduces to

$$\left[T_{k,q} + (\alpha+nq)x^q - prx^{r+q}\right] M_n^{(\alpha)}(x, r, p, k, q)$$
$$= \frac{(n+1)!}{n!} x^q M_{n+1}^{(\alpha)}(x, r, p, k, q),$$

or
$$\left[x^{-q} T_{k,q} + \alpha + nq - prx^r\right] M_n^{(\alpha)}(x, r, p, k, q)$$
$$= (n+1) M_{n+1}^{(\alpha)}(x, r, p, k, q),$$

or
$$\left[k + xD + \alpha + nq - prx^r\right] M_n^{(\alpha)}(x, r, p, k, q) = (n+1) M_{n+1}^{(\alpha)}(x, r, p, k, q),$$

or
$$xD M_n^{(\alpha)}(x, r, p, k, q) + (\alpha+k+nq) M_n^{(\alpha)}(x, r, p, k, q) - prx^r M_n^{(\alpha)}(x, r, p, k, q)$$
$$= (n+1) M_{n+1}^{(\alpha)}(x, r, p, k, q).$$

By making use of (2.6.2), we get

$$prx^r\left[M_n^{(\alpha)}(x, r, p, k, q) - M_n^{(\alpha+r)}(x, r, p, k, q)\right] + (\alpha+k+nq-prx^r) M_n^{(\alpha)}(x, r, p, k, q)$$
$$= (n+1) M_{n+1}^{(\alpha)}(x, r, p, k, q)$$

$$(n+1) M_{n+1}^{(\alpha)}(x, r, p, k, q)$$
$$= (\alpha+k+nq) M_n^{(\alpha)}(x, r, p, k, q) - prx^r M_n^{(\alpha+r)}(x, r, p, k, q),$$

which proves (2.6.10).

2.7 <u>BILATERAL AND BILINEAR GENERATING FUNCTION</u>-

The following theorem holds for

<u>Theorem</u>- If we assume

$$(2.7.1) \quad F[x, t] = \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(x, r, p, k, q),$$

Where $a_n \neq 0$ are arbitrary constant then

$$(2.7.2) \quad (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\left\{1-(1-qt)^{-r/q}\right\}\right] F\left[\frac{x}{(1-qt)^{1/q}}, \frac{yt}{(1-qt)}\right]$$

$$= \sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, b_n(y)\, t^n,$$

(2.7.3) where $$b_n(y) = \sum_{m=0}^{n} a_m \binom{n}{m} (y)^m.$$

Proof - Let

$$F[x, t] = \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(x, r, p, k, q)\, t^n.$$

Replacing t by $tx^q y$, we get

$$F[x, tx^q y] = \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(x, r, p, k, q)\, (tx^q y)^n.$$

Multiplying both sides by $x^\alpha e^{-px^r}$ and then operating with $e^{tT_{k,q}}$, we get

$$e^{tT_{k,q}}\{x^\alpha e^{-px^r} F[x, tx^q y]\}$$
$$= e^{tT_{k,q}} x^\alpha e^{-px^r} \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(x, r, p, k, q)\, (tx^q y)^n.$$

Now

L.H.S

$$= e^{tT_{k,q}}\{x^\alpha e^{-px^r} F[x, tx^q y]\}$$

$$= \frac{x^\alpha}{(1-x^q qt)^{\left(\frac{\alpha+k}{q}\right)}} \exp\left[-p \frac{x^r}{(1-x^q qt)^{r/q}}\right] F\left[\frac{x}{(1-x^q qt)^{1/q}}, \frac{x^q y t}{(1-x^q qt)}\right].$$

(by use of (2.2.6))

Also R.H.S

$$= e^{tT_{k,q}} x^\alpha e^{-px^r} \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(x, r, p, k, q)\, (tx^q y)^n$$

$$= \sum_{n=0}^{\infty} a_n y^n t^n e^{tT_{k,q}}\{x^{\alpha+nq} e^{-px^r} M_n^{(\alpha)}(x, r, p, k, q)\}$$

$$= \sum_{n=0}^{\infty} a_n y^n t^n \sum_{m=0}^{\infty} \frac{t^m}{m!} T_{k,q}^m\{x^{\alpha+nq} e^{-px^r} M_n^{(\alpha)}(x, r, p, k, q)\}$$

$$= \sum_{n=0}^{\infty} a_n y^n t^n \sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{(m+n)!}{n!} x^{\alpha+(m+n)q} e^{-px^r} M_n^{(\alpha)}(x, r, p, k, q)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} y^{n-m} \binom{n}{m} x^{\alpha+nq} e^{-px^r} M_n^{(\alpha)}(x,r,p,k,q)$$

$$= x^{\alpha} e^{-px^r} \sum_{n=0}^{\infty} (x^q t)^n M_n^{(\alpha)}(x,r,p,k,q) \sum_{m=0}^{n} a_{n-m} y^{n-m} \binom{n}{m}$$

$$= x^{\alpha} e^{-px^r} \sum_{n=0}^{\infty} (x^q t)^n M_n^{(\alpha)}(x,r,p,k,q)\, b_n(y),$$

where $\displaystyle b_n(y) = \sum_{m=0}^{n} a_m y^m \binom{n}{m}.$

Therefore,

$$x^{\alpha} (1-x^q q t)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[-p \frac{x^r}{(1-x^q q t)^{r/q}}\right] F\left[\frac{x}{(1-x^q q t)^{1/q}}, \frac{x^q y t}{(1-x^q q t)}\right]$$
$$= x^{\alpha} e^{-px^r} \sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,b,k,q)\, (x^q t)^n\, b_n(y).$$

Replacing t by $x^{-q}t$, we get

$$(1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp[px^r\{1-(1-qt)^{-r/q}\}]\, F\left[\frac{x}{(1-qt)^{1/q}}, \frac{yt}{(1-qt)}\right]$$
$$= \sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,q)\, b_n(y)\, t^n.$$

which is the Bilateral generating function and can be considered as a generalization of well known Hille-Hardy and Weisner's formula.

In particular, when $a_n = 1\Big/\left(\frac{\alpha+k}{q}\right)_n$, then by (3.7.1) we get

$$F[x,t] = \sum_{n=0}^{\infty} \frac{t^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x,r,p,k,q).$$

From equation (2.5.7) we obtain

$$F[x,t] = e^{px^r+qt} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\, {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; -qt\right].$$

In this case (3.7.2) becomes

$$\text{L.H.S} = (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp[px^r\{1-(1-qt)^{-r/q}\}] \exp\left[p \frac{x^r}{(1-qt)^{r/q}}\right] \times$$

$$\times \exp\left[\frac{qyt}{(1-qt)}\right] \sum_{m=0}^{\infty} \frac{\left[-p\frac{x^r}{(1-qt)^{r/q}}\right]^m}{m!} \cdot {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; -\frac{qyt}{(1-qt)}\right]$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r + \frac{qyt}{(1-qt)}\right] \sum_{m=0}^{\infty} \frac{\left[-p\frac{x^r}{(1-qt)^{r/q}}\right]^m}{m!} \cdot {}_1F_1\left[-\frac{mr}{q}, \frac{\alpha+k}{q}; \frac{-qty}{(1-qt)}\right].$$

Also R.H.S. $= \sum_{n=0}^{\infty} M_n^{(\alpha)}(x,r,p,k,q)\, b_n(y)\, t^n,$

where
$$b_n(y) = \sum_{m=0}^{n} \frac{1}{\left(\frac{\alpha+k}{q}\right)_n} \binom{n}{m} y^m$$
$$= \sum_{m=0}^{\infty} \frac{(-n)_m}{\left(\frac{\alpha+k}{q}\right)_m} (-y)^m$$
$$= \frac{n!}{\left(\frac{\alpha+k}{q}\right)_n} L_n^{\left(\frac{\alpha+k-q}{q}\right)}(y).$$

Therefore we get

$$(1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r + \frac{qyt}{(1-qt)}\right] \sum_{m=0}^{\infty} \frac{\left[-p\frac{x^r}{(1-qt)^{r/q}}\right]^m}{m!} \cdot {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; -\frac{qty}{(1-qt)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!}{\left(\frac{\alpha+k}{q}\right)_n} L_n^{\left(\frac{\alpha+k-q}{q}\right)}(y)\, M_n^{(\alpha)}(x,r,p,k,q).$$

Replacing y by $-y$ we get

$$(3.7.4) \quad (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r - \frac{qty}{(1-qt)}\right] \sum_{m=0}^{\infty} \frac{\left[-p\frac{x^r}{(1-qt)^{r/q}}\right]^m}{m!} \cdot {}_1F_1\left[-\frac{mr}{q}; \frac{\alpha+k}{q}; \frac{qty}{(1-qt)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{n!}{\left(\frac{\alpha+k}{q}\right)_n} L_n^{\left(\frac{\alpha+k-q}{q}\right)}(y)\, M_n^{(\alpha)}(x,r,p,k,q).$$

In particular, when $p = q = r = 1$, $k = 0$ and α is replacing by $(\alpha+1)$ in (3.7.4) we obtain the well known Hille-Hardy formula which is given by the relation

$$(1-t)^{-\alpha-1}\exp\left[-\frac{(x+y)}{(1-t)}t\right]{}_0F_1\left[-;\alpha+1;\frac{xyt}{(1-t)^2}\right]$$
$$=\sum_{m=0}^{\infty}\frac{m!}{(\alpha+1)_m}L_m^{(\alpha)}(x)\,L_m^{(\alpha)}(y)\,t^n.$$

On the other hand if we taken $a_n=\left(\frac{c}{q}\right)_n\Big/\left(\frac{\alpha+k}{q}\right)_n$, then by (3.7.1) we get

$$F[x,t]=\sum_{n=0}^{\infty}\frac{(c/q)_n}{\left(\frac{\alpha+k}{q}\right)_n}M_n^{(\alpha)}(x,r,p,k,q).$$

From equation (2.5.8) we obtain

$$F[x,t]=e^{px^r}(1-qt)^{-\frac{c}{q}}\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}.$$
$$\cdot{}_2F_1\left[-\frac{mr}{q},\frac{c}{q};\frac{\alpha+k}{q};\frac{qt}{(qt-1)}\right].$$

In this case (3.7.2) becomes

$$(3.7.5)\quad (1-qt)^{\frac{c}{q}-\left(\frac{\alpha+k}{q}\right)}(1-qt+qty)^{-\frac{c}{q}}e^{px^r}\sum_{m=0}^{\infty}\frac{\left[-p\frac{x^r}{(1-qt)^{r/q}}\right]^m}{m!}.$$
$$\cdot{}_2F_1\left[-\frac{mr}{q},\frac{c}{q};\frac{\alpha+k}{q};\frac{qty}{(1-qt+qty)}\right]$$
$$=\sum_{m=0}^{\infty}{}_2F_1\left[-m,\frac{c}{q};\frac{\alpha+k}{q};y\right]M_m^{(\alpha)}(x,r,p,k,q)\,t^m.$$

In particular when $p=q=r=1$, $k=0$ and replacing α by $(\alpha+1)$ in (3.7.5) we obtain the Weisner's formula which is given by the relation

$$\sum_{m=0}^{\infty}{}_2F_1[-m,c;\alpha+1;y]\,L_m^{(\alpha)}(x)\,t^m$$
$$=(1-t)^{c-\alpha-1}(1-t+yt)^{-c}\exp\left[-\frac{xt}{(1-t)}\right].$$
$$\cdot{}_1F_1\left[\begin{matrix}c;\\ \alpha+1;\end{matrix}\ \frac{xyt}{(1-t)(1-t+yt)}\right].$$

CHAPTER-III

A UNIFIED REPRESENTATION FOR CLASSICAL POLYNOMIALS:

HERMITE, LAGUERRE AND BESSEL POLYNOMIALS

(3.1) INTRODUCTION:

In 1956 Chak [13] introduced two classes of polynomials and studied them separately which are given by

$$G_{n,k}^{(\alpha)} = x^{-\alpha-kn} e^{x} \theta^{n}(x^{\alpha} e^{-x}) \quad , \tag{3.1.1}$$

and

$$P_{n,r}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^{x^{r}} D^{n}(x^{n+\alpha} e^{-x^{r}}), \tag{3.1.2}$$

where $\theta = x^{k+1} D$, $D \equiv \frac{d}{dx}$.

In 1956 F.J. Palas [44] introduced the polynomial T_{kn} by the Rodrigue's formula

$$T_{kn}(x) = \frac{1}{n!} e^{x^{k}} \left(\frac{d}{dx}\right)^{n} (x^{n} e^{-x^{k}}), \tag{3.1.3}$$

was to generalize the work of Steffenson [59] . Toscano [61] Humbert [31] , and Maurice de Duffahel [28] .

At the same time, Raj Gopal [48] studied similar generalisations of Hermite polynomials by replacing r for the exponent 2.

Motivated by the work of Chak [13] , Al-Salam [3] Mittal [41] , Gould-Hopper [29] , Singh-Srivastava [56] , Chatterjea [22] , Bell [6] , Raj Gopal [48] , and Riordan [49] , Joshi and Prajapat [36] considered a class of polynomials $\{M_n^{(\alpha)}(x, r, p, b, k, q) \mid n=0,1,\ldots\}$ defined by

$$(3.1.4) \quad M_n^{(\alpha)}(x, r, p, b, k, q) = C(b,n)\, x^{-\alpha-nq-n} e^{px^r} T_{k,q}^{n}\left(x^{\alpha+bn} e^{-px^r}\right),$$

where $C(b,n)$ is a constant such that

$$C(b,n) = \frac{(-1)^{\frac{n}{2}(b-1)(b-2)}}{2^{\frac{n}{2}b(b-1)}\,(1)_{nb(b-1)}} .$$

b- being a non-negative integer.

The class of polynomials defined by (3.1.4) is an extension of the class of polynomials considered in Chapter II, using the same operator $T_{k,q}$

Evidently following are interesting particular cases.

$$(3.1.5) \quad M_n^{(\alpha)}(x, r, p, 0, 0, -1) = H_n^{(r)}(x, \alpha, p) ,$$

-Gould and Hopper [29].

$$(3.1.6) \quad M_n^{(\alpha-n+1)}(x, r, p, 0, 0, 1) = H_n^{(r)}(x, \alpha, p) ,$$

-Gould and Hopper [29].

$$(3.1.7) \quad M_n^{(\alpha)}(x, r, p, 1, 0, -1) = L_n^{(\alpha)}(x, r, p) ,$$

-Singh and Srivastava [56].

$$(3.1.8) \quad M_n^{(\alpha-n+1)}(x, r, p, 1, 0, 1) = T_{rn}^{(\alpha)}(x, p) ,$$

-Chatterjea [22].

$$(3.1.9) \quad M_n^{(\alpha-2)}(x, -1, p, 2, 0, -1) = \left(\frac{p}{2}\right)^n y_n(x, \alpha, p) ,$$

-Krall and Frink [37].

$$(3.1.10) \quad M_n^{(\alpha-n+1)}(x, -1, p, 2, 0, -1) = \left(\frac{p}{2}\right)^n y_n(x, \alpha, p) ,$$

-Krall and Frink [37].

$$(3.1.11) \quad M_n^{(\alpha-n)}(x, r, p, 1, 0, q) = G_n^{(\alpha)}(x, r, p, q),$$

-Srivastava-Singhal [57]

3.2 <u>THE EXPLICIT FORM</u>- The explicit form for $M_n^{(\alpha)}(x,r,p,b,k,q)$ is given by

(3.2.1) $$M_n^{(\alpha)}(x,r,p,b,k,q) = C(b,n)\, q^n \sum_{m=0}^{n} \frac{(-px^r)^m}{m!}\, x^{(b-1)n} \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j}\left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

To prove this by definition, we have

$$\begin{aligned} M_n^{(\alpha)}(x,r,p,b,k,q) &= C(b,n)\, x^{-\alpha-nq-n} e^{px^r}\, T_{k,q}^n \left[x^{\alpha+bn} e^{-px^r}\right] \\ &= C(b,n)\, x^{-\alpha-nq-n} e^{px^r}\, T_{k,q}^n \left[\sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} x^{\alpha+bn}\right] \\ &= C(b,n)\, x^{-\alpha-nq-n} e^{px^r}\, T_{k,q}^n \left[\sum_{j=0}^{\infty} \frac{(-p)^j}{j!} x^{\alpha+bn+rj}\right] \\ &= C(b,n)\, x^{-\alpha-nq-n} e^{px^r} \left[\sum_{j=0}^{\infty} \frac{(-p)^j}{j!}\, T_{k,q}^n (x^{\alpha+bn+rj})\right] \end{aligned}$$

(3.2.2) $$= C(b,n)\, q^n e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!}\, x^{(b-1)n} \left(\frac{\alpha+k+bn+rj}{q}\right)_n .$$

Further,

$$\begin{aligned} M_n^{(\alpha)}(x,r,p,b,k,q) &= C(b,n)\, q^n \sum_{m=0}^{\infty} \frac{(px^r)^m}{m!} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!}\, x^{(b-1)n} \left(\frac{\alpha+k+bn+rj}{q}\right)_n \\ &= C(b,n)\, q^n \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{j=0}^{m} \frac{(-1)^j (px^r)^m\, x^{(b-1)n}\, m!}{(m-j)!\; j!} \left(\frac{\alpha+k+bn+rj}{q}\right)_n \\ &= C(b,n)\, x^{(b-1)n} q^n \sum_{m=0}^{\infty} \frac{(px^r)^m}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\frac{\alpha+k+bn+rj}{q}\right)_n . \end{aligned}$$

The inner sum is the m^{th} difference of a polynomial of degree n, hence when $m > n$, the inner sum is Zero. Thus

$$M_n^{(\alpha)}(x, r, p, b, k, q)$$

$$= C(b,n)\, q^n x^{(b-1)n} \sum_{m=0}^{n} \frac{(px^r)^m}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

$$= C(b,n)\, q^n x^{(b-1)n} \sum_{m=0}^{n} \frac{(-px^r)^m}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\frac{\alpha+k+bn+rj}{q}\right)_n ;$$

which proves (3.2.1).

In terms of the difference operator $\Delta_{\alpha,r}$ we can have

(3.2.3) $$M_n^{(\alpha)}(x, r, p, b, k, q) = C(b,n)\, q^n x^{(b-1)n} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \Delta^m_{\alpha+k+bn,r} \left(\frac{\alpha+k+bn}{q}\right)_n ,$$

where $\Delta_{\alpha,r} f(\alpha) = f(\alpha+r) - f(\alpha)$.

Equivalently, we can also write

(3.2.4) $$M_n^{(\alpha)}(x, r, p, b, k, q) = C(b,n)\, q^n x^{(b-1)n} \exp[-px^r \Delta_{\alpha+k+bn,r}] \left(\frac{\alpha+k+bn}{q}\right)_n$$

To prove (3.2.3) and (3.2.4) we use well known relation

$$\Delta^m_{\alpha,r} f(\alpha) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(\alpha+rj) ,$$

Now since $\left(\frac{\alpha+k+bn}{q}\right)_n$ is a polynomial of degree n in α, hence the mth difference of this shall be Zero for $m > n$. Hence from (3.2.2) we get (3.2.3)

(3.2.4) is evident from (3.2.3)

From (3.2.2), if $\frac{r}{q} = S$, a positive integer, we have

(3.2.5) $$M_n^{(\alpha)}(x, r, p, b, k, q)$$

$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \left(\frac{\alpha+k+bn}{q}\right)_n {}_S F_S \left[\begin{matrix} \Delta\left(S, \frac{\alpha+k+bn+nq}{q}\right); \\ \Delta\left(S, \frac{\alpha+k+bn}{q}\right); \end{matrix} -px^r\right]$$

Similarly, if $\frac{r}{q} = -S$, S being a positive integer, we get

(3.2.6) $$M_n^{(\alpha)}(x,r,p,b,k,q) = C(b,n)\, q^n x^{(b-1)n} e^{px^r} \left(\frac{\alpha+k+bn}{q}\right)_n {}_sF_s\left[\begin{matrix}\Delta\left(s, \frac{q-(\alpha+k+bn)}{q}\right); \\ \Delta\left(s, \frac{q-nq-(\alpha+k+bn)}{q}\right);\end{matrix} -px^r\right]$$

where $\Delta(s,\alpha)$ stands for S parameters $\frac{\alpha}{s}, \frac{\alpha+1}{s}, \ldots\ldots, \frac{\alpha+s-1}{s}$.

Next, consider the explicit form (3.2.2)

$$M_n^{(\alpha)}(x,r,p,b,k,q) = C(b,n)\, q^n x^{(b-1)n} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \frac{\left(\frac{\alpha+k+bn}{q}\right)_n \left(\frac{\alpha+k+bn}{q}+n\right)_{\frac{rj}{q}}}{\left(\frac{\alpha+k+bn}{q}\right)_{\frac{rj}{q}}}$$

$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \left(\frac{\alpha+k+nb}{q}\right)_n \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \frac{\left(\frac{\alpha+k+bn}{q}+n\right)_{sj}}{\left(\frac{\alpha+k+bn}{q}\right)_{sj}},$$

as $$\left(\frac{\alpha+k+bn+rj}{q}\right)_n = \left(\frac{\alpha+k+bn}{q}\right)_n \left(\frac{\alpha+k+bn+nq}{q}\right)_{sj} \Big/ \left(\frac{\alpha+k+bn}{q}\right)_{sj},$$

since $$(\alpha)_{nk} = k^{nk} \prod_{j=1}^{k} \left(\frac{\alpha+j-1}{k}\right)_n,$$

$$M_n^{(\alpha)}(x,r,p,b,k,q) = C(b,n)\, q^n x^{(b-1)n} e^{px^r} \sum_{j=0}^{\infty} \left(\frac{\alpha+k+bn}{q}\right)_n \frac{\prod_{j=1}^{k}\left(\frac{\alpha+k+bn+nq+j-1}{sq}\right)_n}{\prod_{j=1}^{k}\left(\frac{\alpha+k+bn+j-1}{sq}\right)_n} \frac{(-px^r)^j}{j!}$$

$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \left(\frac{\alpha+k+bn}{q}\right)_n {}_sF_s\left[\begin{matrix}\Delta\left(s, \frac{\alpha+k+bn+nq}{q}\right); \\ \Delta\left(s, \frac{\alpha+k+bn}{q}\right);\end{matrix} -px^r\right],$$

which proves (3.2.5).

Again, if $\frac{r}{q} = -s$, S being positive integer, (3.2.1) reduces to

$$M_n^{(\alpha)}(x,r,p,b,k,q) = C(b,n)\, q^n x^{(b-1)n} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+bn}{q} - sj\right)_n$$

$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \frac{\Gamma\left(\frac{\alpha+k+bn}{q} - sj + n\right)}{\Gamma\left(\frac{\alpha+k+bn}{q} - sj\right)} ,$$

Since $\Gamma(1-\alpha-n) = (-1)^n \dfrac{\Gamma(1-\alpha)}{(\alpha)_n}$,

We have

$$M_n^{(\alpha)}(x,r,p,b,k,q)$$
$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \left(\frac{\alpha+k+bn}{q}\right)_n \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \frac{\left(1-\frac{\alpha+k+bn}{q}\right)_{sj}}{\left(1-\frac{\alpha+k+bn}{q}-n\right)_{sj}}$$
$$= C(b,n)\, q^n x^{(b-1)n} e^{px^r} \left(\frac{\alpha+k+bn}{q}\right)_n {}_sF_s\left[\begin{matrix}\Delta\left(s, \frac{q-\alpha-k-bn}{q}\right); \\ \Delta\left(s, \frac{q-\alpha-k-bn-nq}{q}\right);\end{matrix} -px^r\right],$$

which proves (3.2.6).

If we put b = 1, k = 0 and replacing α by $(\alpha-n)$, in (3.2.5), we have

$$(3.2.7)\quad G_n^{(\alpha)}(x,r,p,q) = \frac{q^n}{n!}\left(\frac{\alpha}{q}\right)_n e^{px^r} {}_sF_s\left[\begin{matrix}\Delta\left(s, \frac{\alpha+nq}{q}\right); \\ \Delta\left(s, \frac{\alpha}{q}\right);\end{matrix} -px^r\right].$$

Again, if we put b = 2, k = 0, q = r = - 1 and replacing α by $(\alpha-2)$ then (3.2.5) changes to

$$y_n(x,\alpha,p) = (-1)^n\left(\frac{x}{p}\right)^n (2-2n-\alpha)_n\, e^{p/x} {}_1F_1\left[\begin{matrix}2-n-\alpha; \\ 2-2n-\alpha;\end{matrix} -\frac{p}{x}\right].$$

using hypergeometric transformation

$${}_1F_1[a; b; z] = e^z {}_1F_1[b-a; b; -z],$$

we get

$$y_n(x;\alpha,p) = (-1)^n\left(\frac{x}{p}\right)^n (2-2n-\alpha)_n\, e^{p/x} e^{-p/x} {}_1F_1\left[-n; 2-2n-\alpha; \frac{p}{x}\right]$$

or

$$(3.2.8)\quad y_n(x;\alpha,p) = \left(\frac{x}{p}\right)^n (\alpha+n-1)_n\, {}_1F_1\left[-n; 2-2n-\alpha; \frac{p}{x}\right].$$

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$$(3.2.9)\quad y_n(x,\alpha,p) = {}_2F_0\left[-n, \alpha+n-1; -; -x/p\right],$$

comparing (3.2.8) and (3.2.9) we have a transformation formula

$$(3.2.10)\quad {}_2F_0\left[-n, \alpha+n-1\,;\,-\,;\,-\frac{x}{p}\right] = \left(\frac{x}{p}\right)^n (\alpha+n-1)_n\ {}_1F_1\left[-n\,;\,2-\alpha-2n\,;\,\frac{p}{x}\right].$$

3.3 <u>THE GENERATING RELATIONS-</u>

For $M_n^{(\alpha)}(x, r, p, b, k, q)$ we have the following generating relations

$$(3.3.1)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{C(b,n)} = e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{\left(\frac{mr+\alpha+k}{q}\right)_{n+\frac{bn}{q}}}{\left(\frac{mr+\alpha+k}{q}\right)_{\frac{bn}{q}}} \frac{\left(q t x^{(b-1)}\right)^n}{n!}.$$

$$(3.3.2)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{C(b,n)} = e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\ {}_{h+1}F_h\left[\begin{matrix}\Delta\left(h+1, \frac{\alpha+k+mr}{q}\right); \\ \Delta\left(h, \frac{\alpha+k+mr}{q}\right);\end{matrix}\ \frac{(h+1)^{(h+1)} q t x^{(b-1)}}{h^h}\right]$$

where $\frac{b}{q} = -h$, h being a positive integer.

$$(3.3.3)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{C(b,n)} = (1+\xi_1)^{-\left(\frac{\alpha+k-q}{q}\right)} \left\{1+(h+1)\xi_1\right\}^{-1} \exp\left[px^r\left\{1-(1+\xi_1)^{-r/q}\right\}\right],$$

where $\quad \xi_1 (1+\xi_1)^h = -q t x^{(b-1)}.$

$$(3.3.4)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{C(b,n)} = e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\ {}_hF_{h-1}\left[\begin{matrix}\Delta\left(h, \frac{q-(\alpha+k+mr)}{q}\right); \\ \Delta\left(h-1, \frac{q-(\alpha+k+mr)}{q}\right);\end{matrix}\ \frac{h^h q t x^{(b-1)}}{(h-1)^{(h-1)}}\right]$$

where $\frac{b}{q} = -h$, h being a positive integer, q is essentially a negative integer and b takes positive values only.

$$(3.3.5)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{C(b,n)} = (1+\xi)^{\left(\frac{\alpha+k}{q}\right)} (1+h\xi)^{-1} \exp\left[px^r\left\{1-(1+\xi)^{r/q}\right\}\right],$$

where $\quad \xi (1+\xi)^{(h-1)} = q t x^{(b-1)}.$

$$(3.3.6)\quad \sum_{n=0}^{\infty} \frac{(\lambda)_n\, M_n^{(\alpha)}(x,r,p,b,k,q)\, t^n}{(\rho)_n\, C(b,n)\; n!}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\; {}_{h+2}F_{h+1}\left[\begin{matrix} \lambda,\ \Delta\left(h+1, \frac{mr+\alpha+k}{q}\right); \\ \rho,\ \Delta\left(h, \frac{mr+\alpha+k}{q}\right); \end{matrix}\ \frac{(h+1)^{(h+1)}\, q t\, x^{(b-1)}}{h^h}\right]$$

$$(3.3.7)\quad \sum_{n=0}^{\infty} \frac{[(\lambda_u)]_n\, M_n^{(\alpha)}(x,r,p,b,k,q)}{[(\rho_v)]_n\; C(b,n)\; n!}\, t^n$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\; {}_{u+h+1}F_{v+h}\left[\begin{matrix} (\lambda_u),\ \Delta\left(h+1, \frac{\alpha+k+mr}{q}\right); \\ (\rho_v),\ \Delta\left(h, \frac{\alpha+k+mr}{q}\right); \end{matrix}\ \frac{(h+1)^{(h+1)}\, q t\, x^{(b-1)}}{h^h}\right]$$

where (λ_u) stands for the set of λ parameters $\lambda_1, \lambda_2, \text{-----}\ \lambda_u$; and (ρ_v) stands for the set of v parameters $\rho_1, \rho_2, \text{-----}\ \rho_v$; also $\Delta(h,\alpha)$ stands for the set of h parameters $\frac{\alpha}{h}, \frac{\alpha+1}{h} \text{----} \frac{\alpha+h-1}{h}$.

$$(3.3.8)\quad \sum_{m=0}^{\infty} \frac{t^m}{m!}\, \frac{M_m^{(\alpha-bm)}(x,r,p,b,k,q)}{C(b,m)}$$

$$= \left[1-qt\,x^{(b-1)}\right]^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\left\{1-\left(1-qt\,x^{(b-1)}\right)^{-r/q}\right\}\right].$$

$$(3.3.9)\quad \sum_{n=0}^{\infty} \frac{[(\lambda_u)]_n\, M_n^{(\alpha-bn)}(x,r,p,b,k,q)\, t^n}{[(\rho_v)]_n\; C(b,n)\; n!}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\; {}_{u+1}F_v\left[\begin{matrix} (\lambda_u),\ \frac{\alpha+k+mr}{q}; \\ \rho_v; \end{matrix}\ q t\, x^{(b-1)}\right].$$

$$(3.3.10)\quad \sum_{n=0}^{\infty} \frac{(\lambda)_n\, M_n^{(\alpha-bn)}(x,r,p,b,k,q)\, t^n}{n!\; C(b,n)\; \left(\frac{\alpha+k}{q}\right)_n}$$

$$= e^{px^r}\left(1-qt\,x^{(b-1)}\right)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\; {}_2F_1\left[\begin{matrix} \lambda,\ -\frac{mr}{q}; \\ \frac{\alpha+k}{q}; \end{matrix}\ \frac{qt\,x^{(b-1)}}{(1-qt\,x^{(b-1)})}\right].$$

<u>Proof of (3.3.1)</u>

Starting from explicit formula (3.2.1)

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \frac{M_n^{(\alpha)}(x,r,p,b,k,q)}{C(b,n)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!}\, q^n x^{(b-1)n} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

$$= \sum_{n=0}^{\infty} \frac{\left(q+x^{(b-1)}\right)^n}{n!} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(px^r)^m}{(m-j)!} \frac{(-1)^j}{j!} \left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

$$= \sum_{n=0}^{\infty} \frac{\left(q+x^{(b-1)}\right)^n}{n!} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(px^r)^{m+j}}{m!} \frac{(-1)^j}{j!} \left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

$$= e^{px^r} \sum_{n=0}^{\infty} \frac{\left(q+x^{(b-1)}\right)^n}{n!} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+bn+rj}{q}\right)_n$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{\left(q+x^{(b-1)}\right)^n}{n!} \frac{\left(\frac{\alpha+k+mr}{q}\right)_{n+\frac{bn}{q}}}{\left(\frac{\alpha+k+mr}{q}\right)_{\frac{bn}{q}}} ,$$

which is (3.3.1).

<u>Proof of (3.3.2)</u>

From (3.3.1), if $\frac{b}{q} = h$, a positive integer, we notice that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{C(b,n)}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha+k+mr}{q}\right)_{(h+1)n}}{\left(\frac{\alpha+k+mr}{q}\right)_{hn}} \frac{\left(q+x^{(b-1)}\right)^n}{n!}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{(h+1)} \left(\frac{\alpha+k+mr+j-1}{(h+1)q}\right)_n}{\prod_{j=1}^{h} \left(\frac{\alpha+k+mr+j-1}{hq}\right)_n} \frac{\left[\frac{(h+1)^{(h+1)} q+x^{(b-1)}}{h^h}\right]^n}{n!} ,$$

where $\Delta(h,\alpha)$ stands for the set of h parameters $\frac{\alpha}{h}, \frac{\alpha+1}{h}, \ldots, \frac{\alpha+h-1}{h}$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_{h+1}F_h \left[\begin{matrix} \Delta\left(h+1, \frac{\alpha+k+mr}{q}\right); \\ \Delta\left(h, \frac{\alpha+k+mr}{q}\right); \end{matrix} \; \frac{(h+1)^{(h+1)} q + x^{(b-1)}}{h^h} \right],$$

which proves (3.3.2).

Proof of (3.3.3)

From Bailey [7] that

(A) $${}_hF_{h-1}\left[\begin{matrix}\Delta(h,d); \\ \Delta(h-1,d);\end{matrix}\; -\frac{x h^h}{(1-x)^h (h-1)^{(h-1)}}\right] = \frac{(1-x)^d}{1+(h-1)x},$$

and $$\frac{x}{(1-x)^h} = \frac{x}{(1-x)}\left(1+\frac{x}{1-x}\right)^h,$$

$$\frac{(1-x)^d}{1+(h-1)} = \frac{\left(1+\frac{x}{1-x}\right)^{1-d}}{\left(1+\frac{hx}{1-x}\right)}.$$

Taking $\xi = \frac{x}{(1-x)}$ and using (A) in (3.3.2), we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x,r,p,b,k,q)}{C(b,n)}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \frac{(1+\xi)^{\left(1-\frac{\alpha+k+mr}{q}\right)}}{(1+h\xi)}$$

$$= e^{px^r}(1+\xi)^{1-\left(\frac{\alpha+k}{q}\right)}\left[1+(h+1)\xi\right]^{-1} \sum_{m=0}^{\infty} \frac{\left[-px^r(1+\xi)^{-r/q}\right]^m}{m!}$$

$$= (1+\xi)^{-\left(\frac{\alpha+k-q}{q}\right)}\left[1+(h+1)\xi\right]^{-1} \exp\left[px^r\left\{1-(1+\xi)^{-r/q}\right\}\right],$$

where $\xi(1+\xi)^h = -q t x^{(b-1)}$,

which is (3.3.3).

In particular when p = q = r = k = 1, and h = 1, $\xi(1+\xi) = -t$ and (3.3.3) reduces to

$$\sum_{n=0}^{\infty} t^n L_n^{(\alpha+n)}(x) = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4t}\right)^{-\alpha}(1-4t)^{-1/2} \exp\left[x\left\{1-\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4t}\right)^{-1}\right\}\right],$$

where $L_n^{(\alpha)}(x)$ denotes generalized Laguerre polynomial.

Proof of (3.3.4):-

From (3.2.1), if $\frac{b}{q} = -h$, h a positive integer, q a negative integer and b takes positive values only, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x,r,P,b,k,q)}{C(b,n)}$$

$$= e^{Px^r} \sum_{m=0}^{\infty} \frac{(-Px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{(q+x^{(b-1)})^n}{n!} \left(\frac{\alpha+k+rm}{q} - hn\right)_n$$

$$= e^{Px^r} \sum_{m=0}^{\infty} \frac{(-Px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{(q+x^{(b-1)})^n}{n!} \frac{\Gamma\left(\frac{\alpha+k+rm}{q} - hn + n\right)}{\Gamma\left(\frac{\alpha+k+rm}{q} - hn\right)}$$

$$= e^{Px^r} \sum_{m=0}^{\infty} \frac{(-Px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{(q+x^{(b-1)})^n}{n!} \frac{h^{hn} \prod_{j=1}^{h} \left[\frac{\left(s - \frac{\alpha+k+mr}{q}\right)}{h}\right]_n}{(h-1)^{n(h-1)} \prod_{j=1}^{(h-1)} \left[\frac{\left(s - \frac{\alpha+k+mr}{q}\right)}{(h-1)}\right]_n}$$

$$= e^{Px^r} \sum_{m=0}^{\infty} \frac{(-Px^r)^m}{m!} {}_hF_{h-1} \left[\begin{matrix} \Delta\left(h, \frac{q-(\alpha+k+mr)}{q}\right); \\ \Delta\left(h-1, \frac{q-(\alpha+k+mr)}{q}\right); \end{matrix} \frac{h^h q + x^{(b-1)}}{(h-1)^{(h-1)}} \right],$$

where $\Delta(h,\alpha)$ stands for h parameter $\frac{\alpha}{h}, \frac{\alpha+1}{h}, \ldots\ldots \frac{\alpha+h-1}{h}$.

This proves (3.3.4).

<u>Proof of (3.3.5):</u>

From (3.3.4), as before, using Bailey formula [7] and taking $\xi = \frac{x}{(1-x)}$, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x,r,P,b,k,q)}{C(b,n)}$$

$$= e^{Px^r} \sum_{m=0}^{\infty} \frac{(-Px^r)^m}{m!} \frac{(1+\xi)^{\left[1-\frac{q-(\alpha+k+mr)}{q}\right]}}{(1+h\xi)}$$

$$= (1+\xi)^{\left(\frac{\alpha+k}{q}\right)} (1+h\xi)^{-1} e^{px^r} \sum_{m=0}^{\infty} \frac{\left[-px^r(1+\xi)^{r/q}\right]^m}{m!}$$

$$= (1+\xi)^{\left(\frac{\alpha+k}{q}\right)} (1+h\xi)^{-1} \exp\left[px^r\left\{1-(1+\xi)^{r/q}\right\}\right],$$

where $\xi(1+\xi)^{(h-1)} = q + x^{(b-1)}$.

Which is (3.3.5).

In particular, put $b=2$, $q=1$, $k=0$ and $r=-1$ so that $h=2$ and $\xi = \left(-1 \pm \sqrt{1-2tx}\right)/2$, and replacing α by $(\alpha-2)$ and t by $\frac{t}{2}$ we get

$$\sum_{n=0}^{\infty} \frac{(pt/2)^n}{n!} y_n(x;\alpha,p) = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-2xt}\right)^{2-\alpha} (1-2xt)^{-1/2} \exp\left[\frac{p}{2x}\left\{1-\sqrt{1-2xt}\right\}\right],$$

where $y_n(x;\alpha,p)$ are Bessel polynomials.

By similar method to that of (3.3.2) we can prove (3.3.6) and (3.3.7).

<u>Proof of (3.3.8)</u>

By explicit formula (3.2.2) replacing α by $(\alpha - bm)$, we have

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{M_m^{(\alpha-bm)}(x,r,p,b,k,q)}{C(b,m)}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{\left(q+x^{(b-1)}\right)^m}{m!} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(\frac{\alpha+k+rj}{q}\right)_n$$

$$= e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \sum_{m=0}^{\infty} \left(\frac{\alpha+k+rj}{q}\right)_m \frac{\left(q+x^{(b-1)}\right)^m}{m!}$$

$$= e^{px^r} \sum_{j=0}^{\infty} \frac{(-px^r)^j}{j!} \left(1-q+x^{(b-1)}\right)^{-\left(\frac{\alpha+k+rj}{q}\right)}$$

$$= e^{px^r} \left(1-q+x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} \sum_{j=0}^{\infty} \frac{\left[-px^r\left(1-q+x^{(b-1)}\right)^{-r/q}\right]^j}{j!}$$

$$= e^{px^r}\left(1-q+x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[-px^r\left(1-q+x^{(b-1)}\right)^{-r/q}\right]$$

$$= \left(1-q+x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\left\{1-\left(1-q+x^{(b-1)}\right)^{-r/q}\right\}\right],$$

which is (3.3.8).

In particular for $b=2$, $k=0$, $q=1$, $r=-1$ and replacing α by $(\alpha-1)$ in (3.3.8)), we get

$$\sum_{m=0}^{\infty} \frac{(pt)^m}{m!} y_m(x, \alpha-m, p) = (1-xt)^{1-\alpha} e^{pt}.$$

<u>Proof of (3.3.9) and (3.3.10)</u>

From the explicit form (3.2.2) replacing α by $(\alpha-bn)$ and sum for n , we get

$$\sum_{n=0}^{\infty} \frac{[(\lambda_\mu)]_n M_n^{(\alpha-bn)}(x, r, p, b, k, q)}{[(\rho_\nu)]_n \; C(b,n) \; n!} t^n$$

$$= e^{px^r} \sum_{n=0}^{\infty} \frac{[(\lambda_\mu)]_n}{[(\rho_\nu)]_n} x^{(b-1)n} q^n \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \left(\frac{\alpha+k+mr}{q}\right)_n \frac{t^n}{n!}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \sum_{n=0}^{\infty} \frac{[(\lambda_\mu)]_n \left(\frac{\alpha+k+mr}{q}\right)_n}{[(\rho_\nu)]_n} \frac{\left(qtx^{(b-1)}\right)^n}{n!}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_{\mu+1}F_{\nu}\left[\begin{matrix}[(\lambda_\mu)], \left(\frac{\alpha+k+mr}{q}\right); \\ [(\rho_\nu)] \quad ;\end{matrix} \; qtx^{(b-1)}\right],$$

which is (3.3.9).

Again, if we put $\lambda_\mu = \rho_\nu = 1$ then (3.3.9) reduces to (3.3.8).

Next, for $\nu = \mu = 1$ in (3.3.9), ~~and~~ $\rho = \frac{\alpha+k}{q}$, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n M_n^{(\alpha-bn)}(x, r, p, b, k, q)\, t^n}{(\rho)_n \; C(b,n) \; n!}$$

$$= e^{px^r} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_2F_1\left[\begin{matrix}\lambda, \frac{\alpha+k+mr}{q}; \\ \frac{\alpha+k}{q} \quad ;\end{matrix} \; qtx^{(b-1)}\right],$$

by using of hypergeometric transformation

$${}_2F_1\left[a,b;c;z\right]=(1-z)^{-a}\,{}_2F_1\left[a,c-b;c;\frac{-z}{1-z}\right],\quad |z|<1,\ \left|\frac{z}{1-z}\right|<1;$$

we get

$$\text{R.H.S} = e^{\rho x^r}\left[1-q+x^{(b-1)}\right]^{-\lambda}\sum_{m=0}^{\infty}\frac{(-\rho x^r)^m}{m!}\,{}_2F_1\left[\begin{matrix}\lambda,-\frac{mr}{q};\\ \frac{\alpha+k}{q};\end{matrix}\ -\frac{q+x^{(b-1)}}{1-q+x^{(b-1)}}\right],$$

which is (3.3.10).

when $k=0$, $q=-1$ and $\lambda=-\alpha$, then (3.3.10) reduces to the result of Joshi [36a]

3.4 A BILINEAR GENERATING FUNCTION-

The following theorem holds for $M_n^{(\alpha)}(x,r,\rho,b,k,q)$:

Theorem - If

(3.4.1) $$F[x,t]=\sum_{m=0}^{\infty}\frac{a_m t^m x^{(1-b)n}}{m!\,C(b,m)}M_m^{(\alpha-bm)}(x,r,\rho,b,k,q),$$

then

(3.4.2) $$\left[1-q+x^{(b-1)}\right]^{-\left(\frac{\alpha+k}{q}\right)}\exp\left[\rho x^r\left\{1-\left(1-q+x^{(b-1)}\right)^{-r/q}\right\}\right].$$
$$\cdot F\left[\frac{x}{(1-q+x^{(b-1)})^{1/q}},\frac{x^{(b-1)}yt}{(1-q+x^{(b-1)})}\right]$$
$$=\sum_{m=0}^{\infty}\gamma_m(y)\frac{t^m}{m!\,C(b,m)}M_m^{(\alpha-bm)}(x,r,\rho,b,k,q),$$

(3.4.3) where $$\gamma_m(y)=\sum_{s=0}^{\infty}\binom{m}{s}a_s y^s.$$

Proof- Let

$$F[x,t]=\sum_{m=0}^{\infty}\frac{a_m t^m x^{(1-b)m}}{m!\,C(b,m)}M_m^{(\alpha-bm)}(x,r,\rho,b,k,q),$$

Replacing t by $tx^q y$, we get

$$F[x,tx^q y]=\sum_{m=0}^{\infty}\frac{a_m x^{(1-b)m}}{m!\,C(b,m)}M_m^{(\alpha-bm)}(x,r,\rho,b,k,q)\,(tx^q y)^m,$$

Multiplying both sides by $x^{\alpha}e^{-\rho x^r}$ and then operating with $e^{tT_{k,q}}$

we get

$$e^{tT_{k,q}}\left\{x^{\alpha}e^{-px^{r}}F[x,\ tx^{q}y]\right\}$$

$$= e^{tT_{k,q}}x^{\alpha}e^{-px^{r}}\sum_{m=0}^{\infty}\frac{a_m}{m!\,C(b,m)}M_m^{(\alpha-bm)}(x,r,p,b,k,q)\,(tx^{q-b+1}y)^m.$$

By use of (2.2.6) , we get

$$\text{L.H.S} = \frac{x^{\alpha}}{(1-qtx^{q})^{\left(\frac{\alpha+r}{q}\right)}}\exp\left[-p\frac{x^{r}}{(1-qtx^{q})^{r/q}}\right].F\left[\frac{x}{(1-qtx^{q})^{1/q}},\ \frac{tx^{q}y}{(1-qtx^{q})}\right].$$

Also R.H.S

$$= \sum_{m=0}^{\infty}\frac{a_m t^m y^m}{C(b,m)\,m!}\,e^{tT_{k,q}}\left\{x^{\alpha+(q-b+1)m}e^{-px^{r}}M_m^{(\alpha-bm)}(x,r,p,b,k,q)\right\}$$

$$= \sum_{m=0}^{\infty}\frac{a_m t^m y^m}{C(b,m)\,m!}\sum_{n=0}^{\infty}\frac{t^n}{n!}T_{k,q}^{n}\left\{x^{\alpha+(q-b+1)m}e^{-px^{r}}M_m^{(\alpha-bm)}(x,r,p,b,k,q)\right\}.$$

From (3.1.4),

$$\text{R.H.S} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m y^m}{m!\,n!}t^{m+n}\,T_{k,q}^{m+n}(x^{\alpha}e^{-px^{r}})$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{a_{m-n}y^{m-n}}{n!\,(m-n)!}t^{m}\,T_{k,q}^{m}(x^{\alpha}e^{-px^{r}})$$

$$= \sum_{m=0}^{\infty}\frac{t^m}{m!}T_{k,q}^{m}(x^{\alpha}e^{-px^{r}})\sum_{n=0}\binom{m}{n}a_{m-n}y^{m-n}$$

$$= \sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{x^{\alpha+(q-b+1)m}}{C(b,m)}e^{-px^{r}}M_m^{(\alpha-bm)}(x,r,p,b,k,q)\sum_{n=0}\binom{m}{n}a_n y^n$$

$$= x^{\alpha}e^{-px^{r}}\sum_{m=0}^{\infty}\frac{(tx^{(q-b+1)})^m}{m!\,C(b,m)}M_m^{(\alpha-bm)}(x,r,p,b,k,q)\,Y_m(y),$$

where $\qquad Y_m(y) = \sum_{n=0}^{m} \binom{m}{n} a_n y^n .$

Therefore, we get

$$\frac{x^{\alpha}}{(1-qtx^{q})^{\left(\frac{\alpha+k}{q}\right)}} \exp\left[-\frac{p x^{r}}{(1-qtx^{q})^{r/q}}\right] . F\left[\frac{x}{(1-qtx^{q})^{1/q}}, \frac{tyx^{q}}{(1-qtx^{q})}\right]$$

$$= x^{\alpha} e^{-px^{r}} \sum_{m=0}^{\infty} \frac{\left(tx^{(q-b+1)}\right)^{m}}{m!\, C(b,m)} M_m^{(\alpha-bm)}(x,r,p,b,k,q)\, Y_m(y).$$

Replacing t by t/x^{q-b+1}, we get

$$\left(1-qtx^{(b-1)}\right) \exp\left[px^{r}\left\{1-\left(1-qtx^{(b-1)}\right)\right\}\right] . F\left[\frac{x}{(1-qtx^{(b-1)})^{1/q}}, \frac{x^{(b-1)}yt}{(1-qtx^{(b-1)})}\right]$$

$$= \sum_{m=0}^{\infty} \frac{t^{m}}{m!\, C(b,m)} M_m^{(\alpha-bm)}(x,r,p,b,k,q)\, Y_m(y),$$

which is the required result.

We list below some applications of this theorem.

<u>COROLLARY 1:</u>

For $b=q=1$, $k=0$ and replacing α by $(\alpha+1)$, in (3.4.2), we get

$$(1-t)^{-(\alpha+1)} \exp\left[px^{r}\left\{1-(1-t)^{-r/q}\right\}\right] F\left[\frac{x}{(1-t)}, \frac{yt}{(1-t)}\right]$$

$$= \sum_{m=0}^{\infty} Y_m(y) \frac{t^{m}}{m!} L_m^{(\alpha)}(x,r,p) .$$

<u>COROLLARY 2:</u>

For $b=2$, $k=0$, $q=1$, $r=-1$ and replacing α by $(\alpha-1)$ in (3.4.2), we get

$$x^{-\alpha}(x-t)^{\alpha} \exp\left[px^{r}\left\{1-x^{-r}(x-t)^{r}\right\}\right] F\left[(x-t), \frac{yt}{(x-t)}\right]$$

$$= \sum_{m=0}^{\infty} Y_m(y) \frac{t^{m}}{m!} H_m^{(r)}(x,\alpha,p) .$$

COROLLARY 3

For $b=2, k=0, q=1, \gamma=-1$ and replacing α by $(\alpha-1)$ in (3.4.2), we get

$$(1-tx)^{-(\alpha-1)} e^{\rho t} F\left[\frac{x}{(1-xt)}, \frac{xyt}{(1-xt)}\right] = \sum_{m=0}^{\infty} \gamma_m(y) \frac{(\rho t)^m}{m!} y_m(x, \alpha-m, \rho).$$

COROLLARY 4

Let $a_m = [(\lambda_u)]_m / [(\rho_v)]_m$ in (3.4.2) then (3.4.1) reduces to

$$F[x,t] = \sum_{m=0}^{\infty} \frac{[(\lambda_u)]_m}{[(\rho_v)]_m} \frac{t^m x^{(1-b)m}}{m!\, C(b,m)} M_m^{(\alpha-bm)}(x,\gamma,\rho,b,k,q),$$

Also R.H.S. of (3.4.2)

$$= \left(1-qt x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[\rho x^{\gamma}\left\{1-\left(1-qt x^{(b-1)}\right)^{-\gamma/q}\right\}\right].$$

$$\cdot \sum_{m=0}^{\infty} \frac{[(\lambda_u)]_m M_m^{(\alpha-bm)}\left[\frac{x}{(1-qt x^{(b-1)})^{1/q}}, \gamma, \rho, b, k, q\right]}{[(\rho_v)]_m\, C(b,m)\, m!} \cdot \left[\frac{x}{(1-qt x^{(b-1)})^{1/q}}\right]^{(1-b)m} \cdot \left[\frac{x^{(b-1)} y t}{(1-qt x^{(b-1)})}\right]^m.$$

Using (3.2.2), we get

$$= \left(1-qt x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} e^{\rho x^{\gamma}} \sum_{m=0}^{\infty} \frac{[(\lambda_u)]_m}{[(\rho_v)]_m} \frac{\left[\frac{qt x^{(b-1)}}{(1-qt x^{(b-1)})}\right]^m}{m!}$$

$$\cdot \sum_{j=0}^{\infty} \frac{\left[-\rho \frac{x^{\gamma}}{(1-qt x^{(b-1)})^{\gamma/q}}\right]^j}{j!} \left(\frac{\alpha+k+\gamma j}{q}\right)_m$$

$$= \left(1-qt x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} e^{\rho x^{\gamma}} \sum_{j=0}^{\infty} \frac{\left[-\rho \frac{x^{\gamma}}{(1-qt x^{(b-1)})^{\gamma/q}}\right]^j}{j!}.$$

$$\cdot \sum_{m=0}^{\infty} \frac{[(\lambda_u)]_m}{[(\rho_v)]_m} \left(\frac{\alpha+k+\gamma j}{q}\right)_m \frac{\left[\frac{qt y x^{(b-1)}}{(1-qt x^{(b-1)})}\right]^m}{m!}$$

$$= \left(1-qt\,x^{(b-1)}\right)^{-\frac{\alpha+k}{q}} e^{\rho x^{r}} \sum_{j=0}^{\infty} \frac{\left[-\rho \dfrac{x^{r}}{(1-qtx^{(b-1)})^{r/q}}\right]^{j}}{j!} .$$

$$\cdot\, {}_{u+1}F_{v}\left[\lambda u,\ \frac{\alpha+k+mr}{q};\ \rho_v;\ \frac{qty\,x^{(b-1)}}{(1-qtx^{(b-1)})}\right],$$

and L.H.S. of (3.4.2)

$$= \sum_{m=0}^{\infty} \frac{t^{m} x^{(1-b)m}}{m!\; C(b,m)} M_{m}^{(\alpha-bm)}(x,r,\rho,b,k,q).$$

$$\cdot \sum_{n=0}^{\infty} \frac{m!}{n!\,(m-n)!} \frac{[(\lambda u)]_n}{[(\rho_v)]_n} y^{n}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m} x^{(1-b)m}}{m!\; C(b,m)} M_{m}^{(\alpha-bm)}(x,r,\rho,b,k,q).$$

$$\cdot \sum_{n=0}^{m} \frac{(-m)_n [(\lambda u)]_n}{[(\rho_v)]_n} \frac{(-y)^{n}}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{t^{m} x^{(1-b)m}}{m!\; C(b,m)} M_{m}^{(\alpha-bm)}(x,r,\rho,b,k,q).$$

$$\cdot\, {}_{u+1}F_{v}\left[\begin{matrix} -m,\ (\lambda u); \\ (\rho_v); \end{matrix}\ -y\right].$$

Therefore ,

$$\left(1-qt\,x^{(b-1)}\right)^{-\left(\frac{\alpha+k}{q}\right)} e^{\rho x^{r}} \sum_{j=0}^{\infty} \frac{\left[-\rho \dfrac{x^{r}}{(1-qtx^{(b-1)})^{r/q}}\right]^{j}}{j!} .$$

$$\cdot\, {}_{u+1}F_{v}\left[\begin{matrix} \lambda u,\ \dfrac{\alpha+mr+k}{q}; \\ \rho_v\ ; \end{matrix}\ \frac{qty\,x^{(b-1)}}{(1-qtx^{(b-1)})}\right]$$

$$= \sum_{m=0}^{\infty} \frac{t^{m} x^{(1-b)m}}{m!\; C(b,m)} M_{m}^{(\alpha-bm)}(x,r,\rho,b,k,q) \cdot {}_{u+1}F_{v}\left[\begin{matrix} -m,\ (\lambda u); \\ (\rho_v); \end{matrix}\ -y\right].$$

This generalizes the well known Hille-Hardy and Weisner's formula .

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6)

THE OPERATOR $T_{k,q}$ AND CHARACTERIZATION OF A CLASS OF POLYNOMIALS BY THE GENERALIZED RODRIGUE'S FORMULA

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1. Introduction

Employing the operator x^2D where $D=\frac{d}{dx}$ Chak [2] defined the generalized Laguerre polynomials by means of

$$L_n^{(\alpha)}(x)=x^{-\alpha-n-1}\,e^{x}(x^2D)^n\,(x^{\alpha+1}\,e^{-x}) \tag{1.1}$$

Later, Al-Salam [1] characterized these polynomials in terms of the operator $\theta=x(1+xD)$ and proved that

$$\theta^n\,x^{\alpha}\,e^{-x}=x^{\alpha+n}\,e^{-x}\,n!\,L_n^{(\alpha)}(x) \tag{1.2}$$

Recently, Mittal [7] observed that relations (1.1) and (1.2) can, in fact, be derived from a more general operational representation. To this end he considered the operator $T_k=x(k+xD)$, k being constant and showed that the polynomial set $\{T_{\nu n}^{(\alpha)}(x)\,|\,n=0, 1, 2, \cdots\}$ [6] where

$$T_{\nu n}^{(\alpha)}(x)=\frac{1}{n!}x^{-\alpha}\,e^{p_\nu(x)}\,D^n\,[x^{\alpha+n}\,e^{-p_\nu(x)}] \tag{1.3}$$

$p_\nu(x)$ is a polynomial in x of degree r, admit the relationship

$$T_{\nu n}^{(\alpha+k-1)}(x)=\frac{1}{n!}\,x^{-\alpha-n}\,e^{p_\nu(x)}\,T_k^n[x^{\alpha}\,e^{-p_\nu(x)}] \tag{1.4}$$

in terms of the operator T_k.

The question, that attracted our attention, was if these aforementioned characterizations could be unified. This led us [5] to define the operator $T_{k,q}=x^q(k+xD)$ and the introduction of the polynomial set in the form

$$M_{\nu n}^{(\alpha)}(x, k, q)=\frac{1}{n!}\,x^{-\alpha-nq}\,e^{p_\nu(x)}\,T_{k,q}^n\,[x^{\alpha}\,e^{-p_\nu(x)}] \tag{1.5}$$

where $p_\nu(x)$ is a polynomial in x of degree r and k and q are constants.

In so far as the generality is concerned, obviously the definition is quite general. Indeed, it provided a direct generalization of all the known generalizations of classical Laguerre polynomials for which one may refer to the work of

Chatterjea [3] and Singh and Srivastava [9]. Yet the definition is limited, since it is not possible to carry over a number of well known properties to the generalized case. This limitation is, to a great extent, overcome by introducing the polynomial set $\{M_n^{(\alpha)}(x, r, p, k, q)\}, n=0, 1, 2, \cdots\}$,

$$M_n^{(\alpha)}(x, r, p, k, q)=\frac{1}{n!}\; x^{-\alpha-nq}\, e^{px^r}\; T_{k,q}^n\,(x^\alpha e^{-px^r}) \tag{1.6}$$

where p, r, k and q are constants and assume integral values.

For $k=0$, one would obtain the polynomial set $\{G_n^{(\alpha)}(x, r, p, q)\}, n=0, 1, 2, \cdots\}$ considered earlier by Srivastava and Singhal [10]. It may be noted here that the polynomial set $\{G_n^{(\alpha)}(x, r, p, q)\}$ can not be said to give a direct generalization of the classical Hermite or the generalized Hermite polynomials of Gould and Hopper [4], since

$$G_n^{(0)}(x, 2, 1, -1)=\frac{(-x)^n}{n!}\; H_n(x),$$

and

$$G_n^{(\alpha)}(x, r, p, -1)=\frac{(-x)^n}{n!}\; H_n^r(x, \alpha, p).$$

The question, therefore, naturally arises if there does in fact exist a unified representation for the two classes of seemingly alike polynomials, the Laguerre and Hermite polynomials. Interestingly, the answer is in the affirmative and will be presented in detail in a subsequent communication.

2. The operator $T_{k,q}$

We [5] have defined the operator $T_{k,q}$ as

$$T_{k,q}\equiv x^q\,(k+xD), \text{ where } D\equiv\frac{d}{dx}.$$

Listed below are some of the properties that we shall require in our investigations:

$$T_{k,q}^n\,(x^{\alpha+m})=q^n\Big(\frac{\alpha+m+k}{q}\Big)_n x^{\alpha+m+nq} \tag{2.1}$$

where as usual $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1),\; n\geqslant 1,\; (\alpha)_0=1.$

$$F(T_{k,q})\{x^\alpha f(x)\}=x^\alpha F(T_{k,q}+x^q\alpha)f(x) \tag{2.2}$$

$$F(T_{k,q})\{e^{g(x)} f(x)\}=e^{g(x)}\, F\{T_{k,q}+x^{q+1}\, g'(x)\}f(x) \tag{2.3}$$

$$(T_{k,q})^n(xuv)=x\sum_{m=0}^{n}\binom{n}{m}(T_{k,q}^{n-m}\, v)\,(T_{1,q}^m\, u) \tag{2.4}$$

where $T_{1,\,q}=x^q(1+xD)$

In particular,

$$(T_{k,\,q})^n\,(uv)=\sum_{m=0}^{n}\binom{n}{m}(T_{k,\,q}^{n-m}v)(T_q^m u),\ T_q=x^{q+1}\,D \tag{2.5}$$

$$e^{tT_{k,q}}\,[x^\alpha f(x)]=\frac{x^\alpha}{(1-x^q qt)^{\frac{\alpha+k}{q}}}f\left[\frac{x}{(1-x^q qt)^{1/q}}\right] \tag{2.6}$$

$$_{\lambda}F_{\mu}\left[\begin{matrix}(a_\lambda);\\(b_\mu);\end{matrix}\ tT_{k,q}\right]x^\alpha e^{px^r}=\sum_{j=0}^{\infty}\frac{(p)^j}{j!}\,x^{\alpha+rj}$$

$$_{\lambda+1}F_{\mu}\left[\begin{matrix}(a_\lambda),\ \left(\frac{\alpha+rj+k}{q}\right);\\(b_\mu);\end{matrix}\quad x^q qt\right] \tag{2.7}$$

where (a_λ) stands for the sequence of λ parameters namely $a_1,\,a_2,\,\cdots a_\lambda$ with similar interpretation for (b_μ).

In particular

$$_0F_1\left[-;\ \frac{\alpha+k}{q};\ tT_{k,\,q}\right]x^\alpha e^{-px^r}=x^\alpha\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}\,{}_1F_1\left[\frac{\alpha+k+mr}{q};\ \frac{\alpha+k}{q};x^q qt\right] \tag{2.8}$$

In addition, note also that

$$\prod_{j=0}^{n-1}(\delta+\alpha+k-prx^r+jq).\,1=n!\ M_n^{(\alpha)}(x,\,r,\,p,\,k,\,q) \tag{2.9}$$

which can be put in an equivalent form

$$M_n^{(\alpha)}(x,\,r,\,p,\,k,\,q)=\frac{q^n}{n!}\,e^{px^r}\left(\frac{\delta+\alpha+k}{q}\right)_n e^{-px^r},\ \delta=xD, \tag{2.10}$$

and suggests the elegant operational relationship

$$x^{-nq}\,T_{k,\,q}^n=(\delta+k)(\delta+k+q)\cdots(\delta+k+\overline{n-1}q). \tag{2.11}$$

3. The explicit form

It follows from (1.6) and (2.1) that

$$M_n^{(\alpha)}(x,\,r,\,p,\,k,\,q)=\frac{q^n}{n!}\sum_{m=0}^{n}\frac{(-px^r)^m}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{\alpha+rj+k}{q}\right)_n \tag{3.1}$$

This can be put in the form

$$M_n^{(\alpha)}(x,\,r,\,p,\,k,\,q)\quad\frac{q^n}{n!}\,e^{px^r}\,(a)_n\,y \tag{3.2}$$

$$M_n^{\alpha}(x, r, k, q) = \frac{q^n}{n!} e^{px^r} \sum_{j=0}^{\infty} \frac{(-p)^j \cancel{(\alpha+k)} \left(\frac{\alpha+k}{q}+n\right)_{mj}}{j! \left(\frac{\alpha+k}{q}\right)_{mj}} x^{rj}$$

where $y = \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} \frac{(a+n)_{mj}}{(a)_{mj}} x^{rj}$, $\frac{\alpha+k}{q} = a$,

$\frac{r}{q} = m$, m being a positive integer.

Now, if

$$\Delta_{\alpha, r} f(\alpha) = f(\alpha + r) - f(\alpha)$$

so that

$$\Delta^m_{\alpha, r} f(\alpha) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(\alpha + rj), \tag{3.3}$$

it becomes obvious that the inner series in (3.1) can be expressed as the m^{th} difference of a polynomial of degree n in α which vanishes for $m > n$, such that

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n}{n!} \sum_{m=0}^{n} \frac{(-px^r)^m}{m!} \Delta^m_{\alpha+k, r} \left(\frac{\alpha+k}{q}\right)_n \tag{3.4}$$

This suggests on the one hand that $M_n^{(\alpha)}(x, r, p, k, q)$ is a polynomial of degree n in x^r, since one could write

$$M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n}{n!} \exp\left[-px^r \Delta_{\alpha+k, r}\right] \left(\frac{\alpha+k}{q}\right)_n \tag{3.5}$$

whereas in view of the definition and the formula (2.5) $M_n^{(\alpha)}(x, r, p, k, q)$ can be expressed as a polynomial of degree n in α in the form

$$M_n^{(\alpha)}(x, r, p, k, q) = \sum_{m=0}^{n} \frac{q^m}{m!} \left(\frac{\alpha}{q}\right)_m M^{(0)}_{n-m}(x, r, p, k, q) \tag{3.6}$$

4. The differential equation

Assuming that $\frac{r}{q} = m$, a positive integer and employing the operator $T_{k,q}$, possessing the property that

$$x^{-q}\, T_{k,q}\, x^n = (n+k) x^n,$$

in view of (3.2), we obtain

$$[x^{-q} T_{k,q} - k] \left[\frac{m}{r}(x^{-q} T_{k,q} - k) + a - m\right]_m y$$

$$= r \sum_{j=0}^{\infty} \frac{(-p)^{j+1} (a+n)_{mj+m}\, x^{rj+r}}{j! (a)_{mj}}$$

$$= -prx^r \left[\frac{m}{r}\{x^{-q} T_{k,q} - k\} + a + n\right]_m y.$$

This shows that the polynomials satisfy the differential equation

$$\Big[(x^{-q}T_{k,q}-k-prx^r)\Big\{\frac{m}{r}(x^{-q}T_{k,q}-k)-prx^r+a-m\Big\}_m$$
$$+prx^r\Big\{\frac{m}{r}(x^{-q}T_{k,q}-k)-prx^r+a+n\Big\}_m\Big]M_n^{(\alpha)}(x,r,p,k,q)=0 \qquad (4.1)$$

Or, since (2.11) holds, we have alternatively

$$\Big[(\delta-prx^r)\Big(\frac{m}{r}\delta-pmx^r+a-m\Big)_m+prx^r\Big(\frac{m}{r}\delta-pmx^r$$
$$+a+n\Big)_m\Big]M_n^{(\alpha)}(x,r,p,k,q)=0, \qquad (4.2)$$

giving rise to the product form

$$\Big[(\delta-prx^r)\prod_{j=1}^{m}(\delta-prx^r+\alpha+k-r+jq-q)$$
$$+prx^r\prod_{j=1}^{m}(\delta-prx^r+\alpha+k+nq+jq-q)\Big]M_n^{(\alpha)}(x,r,p,k,q)=0. \qquad (4.3)$$

If, however, $\frac{r}{q}=-m$, where m is a positive integer, the differential equation assumes the form

$$\Big[(\delta-prx^r)\prod_{j=1}^{m}(\delta-prx^r-\alpha-k-r-nq+jq)$$
$$+prx^r\prod_{j=1}^{m}(\delta-prx^r-\alpha-k+jq)\Big]M_n^{(\alpha)}(x,r,p,k,q)=0 \qquad (4.4)$$

5. Generating functions

For the sake of brevity, if we assume that $[(a_\lambda)]_n$ and $[(b_\mu)]_n$ stand for $\prod_{j=1}^{\lambda}(a_j)_n$ and $\prod_{j=1}^{\mu}(b_j)_n$ respectively, then it follows from (1.6) that

$$\sum_{n=0}^{\infty}\frac{[(a_\lambda)]_n}{[(b_\mu)]_n}M_n^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$=e^{px^r}x^{-\alpha}\,{}_\lambda F_\mu\left[\begin{matrix}(a_\lambda);\\(b_\mu);\end{matrix}\;x^{-q}tT_{k,q}\right]x^\alpha e^{-px^r}$$

Therefore, by virtue of (2.7), one obtains the generating relation

$$\sum_{n=0}^{\infty}\frac{[(a_\lambda)]_n}{[(b_\mu)]_n}M_n^{(\alpha)}(x,r,p,k,q)\,t^n$$
$$=e^{px^r}\sum_{n=0}^{\infty}\frac{(-px^r)^n}{n!}\,{}_{\lambda+1}F_\mu\left[\begin{matrix}(a_\lambda),\Big(\frac{\alpha+k+nr}{q}\Big);\\(b_\mu);\end{matrix}\;qt\right] \qquad (5.1)$$

In particular, for $\lambda=\mu$, $a_j=b_j$, $j=1, 2, \cdots, \lambda$ (or μ), (5.1) reduces to

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, t^n = (1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\{1-(1-qt)^{-\frac{r}{q}}\right] \tag{5.2}$$

Next in (3.1), if we replace α by α-nq, multiply both the sides by t^n and sum for $n>0$, we obtain

$$\sum_{n=0}^{\infty} M_n^{(\alpha-nq)}(x, r, p, k, q)\, t^n = (1+qt)^{\frac{\alpha+k-q}{q}} \exp\left[px^r\{1-(1+qt)^{\frac{r}{q}}\}\right] \tag{5.3}$$

Multiplication of this by $x^{\alpha} e^{-pxr}$ and then operation by $T_{k,q}^m$ yields

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha-nq)}(x, r, p, k, q)\, t^m = (1+qt)^{\frac{\alpha+k-q}{q}} \cdot \exp\left[px^r\{1-(1+qt)^{\frac{r}{q}}\}\right]$$

$$\cdot M_n^{(\alpha)}\left[\frac{x}{(1+qt)^{\frac{1}{q}}}, r, p, k, q\right] \tag{5.4}$$

On the other hand, since

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$= x^{\alpha-mq} e^{px^r} e^{x^{-\alpha}} T_{k,q}[x^{\alpha+mq} e^{-px^r} M_m^{(\alpha)}(x, r, p, k, q)], \tag{5.5}$$

by an appeal to (2.6), we get

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{m+n}^{(\alpha)}(x, r, p, k, q)\, t^n$$

$$= (1-qt)^{-\left(\frac{\alpha+k}{q}\right)-m} \cdot \exp\left[px^r\{1-(1-qt)^{-\frac{r}{q}}\}\right] M_m^{(\alpha)}\left[\frac{x}{(1-qt)^{\frac{1}{q}}}, r, p, k, q\right] \tag{5.6}$$

where, $m=0, 1, 2, \cdots$.

Interestingly, for $m=0$, (5,4) and (5.6) reduce to (5.3) and (5.2) respectively. Again, by definition

$$\sum_{n=0}^{\infty} \frac{(x^q t)^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q) = x^{-\alpha} e^{px^r} {}_0F_1\left[-;\ \frac{\alpha+k}{q};\ tT_{k,q}\right] x^{\alpha} e^{-px^r} \tag{5.7}$$

Hence by making use of (2.8), it simplifies to

$$\sum_{n=0}^{\infty} \frac{t^n}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q) = e^{px^r+qt} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!}\, {}_1F_1\left[-\frac{mr}{q};\ \frac{\alpha+k}{q};\ -qt\right] \tag{5.8}$$

We also notice from (2.8) that if $\frac{r}{q}=s$, s is a positive integer, then

$$
{}_0F_1\left[-;\ \frac{\alpha+k}{q};\ tT_{k,q}\right]x^\alpha e^{-px^r}
$$

$$
= x^\alpha \sum_{n=0}^{\infty} \frac{(x^r q t)^n}{n!}\, {}_sF_s\left[\begin{matrix}\Delta\left(S,\ \dfrac{\alpha+k+nq}{q}\right); \\ \Delta\left(S,\ \dfrac{\alpha+k}{q}\right);\end{matrix}\quad -px^r\right] \tag{5.9}
$$

where $\Delta(S, \alpha)$ stands for the set of s parameters

$$
\frac{\alpha}{S},\ \frac{\alpha+1}{S},\ \ldots,\ \frac{\alpha+S-1}{S}.
$$

This on comparison with (5.7) yields another form of the explicit formula

$$
M_n^{(\alpha)}(x, r, p, k, q) = \frac{q^n\left(\frac{\alpha+k}{q}\right)_n}{n!}\, e^{px^r}\, {}_sF_s\left[\begin{matrix}\Delta\left(S,\ \dfrac{\alpha+k+nq}{q}\right); \\ \Delta\left(S,\ \dfrac{\alpha+k}{q}\right);\end{matrix}\quad -px^r\right] \tag{5.10}
$$

Next, by making an appeal to (2.10), we obtain

$$
\sum_{n=0}^{\infty} \frac{\left(\frac{c}{q}\right)_m}{\left(\frac{\alpha+k}{q}\right)_n} M_n^{(\alpha)}(x, r, p, k, q)\, t^n
$$

$$
= e^{px^r}(1-qt)^{-\frac{c}{q}} \sum_{m=0}^{\infty} \frac{(px^r)^m}{m!}\, {}_2F_1\left[-\frac{m}{q},\ \frac{c}{q};\ \frac{\alpha+k}{q};\ \frac{qt}{qt-1}\right] \tag{5.11}
$$

If we put $\lambda=\mu=1$, $c_1=\frac{c}{q}$ and $_1=\frac{\alpha+k}{q}$, then (5.1) also leads us to (5.11).

6. Some applications of generating functions

Observe that the generating relations (5.2) and (5.3) on comparison, yield the recursion formula

$$
M_n^{(\alpha)}(x, r, p, k, -q) = M_n^{(\alpha-nq)}(x, r, p, k, q) + qM_{n-1}^{(\alpha-nq+q)}(x, r, p, k, q) \tag{6.1}
$$

Two other immediate consequences of (5.2) are

$$
DM_n^{(\alpha)}(x, r, p, k, q) = prx^{r-1}\left[M_n^{(\alpha)}(x, r, p, k, q) - M_n^{(\alpha+r)}(x, r, p, k, q)\right] \tag{6.2}
$$

and

$$
(\alpha+k-q)M_n^{(\alpha+k)}(x, r, p, k, q) = prx^r M_n^{(\alpha+k+r)}(x, r, p, k, q)
$$

$$
+(n+1)M_{n+1}^{(\alpha-q)}(x, r, p, k, q) \tag{6.3}
$$

Further, in view of the generating relation (5.2), we readily establish the following multiplication, addition and the summation formulas:

$$M_n^{(\alpha)}(x,\ r,\ mp,\ k,\ q) = M_n^{(\alpha)}(m^{\frac{1}{r}}x,\ r,\ p,\ k,\ q) \tag{6.4}$$

$$M_n^{\left(\sum_{s=1}^{m}\alpha_s\right)}\left[x,\ r,\ \sum_{s=1}^{m} p_s,\ mk,\ q\right] = \sum_{i_1+\cdots+i_m=n} \prod_{j=1}^{m} M_{i_j}^{(\alpha_j)}(x,\ r,\ p_j,\ k,\ q) \tag{6.5}$$

$$M_n^{\left(\sum_{s=1}^{m}\alpha_s\right)}\left[\left(\sum_{s=1}^{m} x_s\right)^{\frac{1}{r}},\ r,\ p,\ mk,\ q\right] = \sum_{i_1+\cdots+i_m=n} \prod_{j=1}^{m} M_{i_j}^{(\alpha_j)}\left[(x_j^{\frac{1}{r}})\ r,\ p,\ k,\ q\right] \tag{6.6}$$

and

$$M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q) = \sum_{m=0}^{n} \frac{\left(\frac{\alpha-\beta}{q}\right)_m}{m!}\, q^m M_{n-m}^{(\beta)}(x,\ r,\ p,\ k,\ q) \tag{6.7}$$

Note also from definition that

$$T_{k,q}^m\,[e^{-px^r}x^{\alpha+nq}M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)]$$

$$= \frac{(m+n)!}{n!}\, x^{\alpha+\overline{m+n}\,q}\, e^{-px^r} M_{m+n}^{(\alpha)}(x,\ r,\ p,\ k,\ q) \tag{6.8}$$

Therefore by an appeal to (2.2) and (2.3), one obtains

$$[T_{k,q}+(\alpha+nq)\,x^q-prx^{r+q}]^m\ M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)$$

$$= \frac{(m+n)!}{n!}\, x^{mq}\ M_{m+n}^{(\alpha)}(x,\ r,\ p,\ k,\ q). \tag{6.9}$$

From $m=1$, it reduces to

$$[xD+\alpha+k+nq-prx^r]\ M_n^{(\alpha)}(x,r,p,k,q) = (n+1)M_{n+1}^{(\alpha)}(x,r,p,k,q) \tag{6.10}$$

Obviously, elimination of $DM_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)$ between (6.2) and (6.10) would lead to the recurrence relation

$$(n+1)\,M_{n+1}^{(\alpha)}(x,r,p,k,q) = (\alpha+k+nq)M_n^{(\alpha)}(x,r,p,k,q)$$

$$-prx^r M_n^{(\alpha+r)}(x,r,p,k,q) \tag{6,11}$$

7. Bilateral and bilinear generating function

THEOREM. *If we assume*

$$F[x,t] = \sum_{n=0}^{\infty} a_n M_n^{(\alpha)}(x,\ r,\ p,\ k,\ q)\, t^n,\quad a_n \neq 0 \text{ are arbitrary} \tag{7.1}$$

constants, then

$$(1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r\{1-(1-qt)^{-\frac{r}{q}}\}\right] F\left[\frac{x}{(1-qt)^{\frac{1}{q}}},\ \frac{yt}{1-qt}\right] \tag{7.2}$$

$$= \sum_{n=0}^{\infty} M_n^{(\alpha)}(x, r, p, k, q)\, b_n(y)\, t^n$$

where, $$b_n(y) = \sum_{m=0}^{n} a_m \binom{n}{m} (y)^m. \tag{7.3}$$

To prove (7.2), replace t by $tx^q y$, multiply both the sides by $x^\alpha e^{-px^r}$ and then operate by $e^{tT_{k,q}}$. By a simple change of variable and in view of the formulas (2.6) and (6.8), the bilateral generating relation (7.2) is established.

As an application of the above theorem we demonstrate how the well known Hille-Hardy formula [8, p212]

$$(1-t)^{-\alpha-1} e^{-\left(\frac{x+y}{1-t}\right)t} {}_0F_1\left[-;\alpha+1;\frac{xyt}{(1-t)^2}\right] = \sum_{m=0}^{\infty} \frac{m!}{(\alpha+1)_m} L_m^{(\alpha)}(x) L_m^{(\alpha)}(y) t^m \tag{7.4}$$

and Weisner's formula [8, p. 213]

$$\sum_{m=0}^{\infty} {}_2F_1\left[-m, c; \alpha+1; y\right] L_m^{(\alpha)}(x) t^m = (1-t)^{c-\alpha-1}(1-t+yt)^{-c} e^{\frac{-xt}{1-t}} \cdot$$

$${}_1F_1\left[c; \alpha+1; \frac{xyt}{(1-t)(1-t+yt)}\right] \tag{7.5}$$

can be obtained.

First assume $a_n = \dfrac{1}{\left(\frac{\alpha+k}{q}\right)_n}$, then from (5.8)

$$F[x, t] = e^{px^r+qt} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} {}_1F_1\left[-\frac{mr}{q};\ \frac{\alpha+k}{q};\ -qt\right]$$

so that (7.2) yields the interesting bilinear generating function

$$(1-qt)^{-\left(\frac{\alpha+k}{q}\right)} \exp\left[px^r - \frac{qty}{1-qt}\right] \sum_{m=0}^{\infty} \frac{\left[-p\left\{\frac{x}{(1-qt)^{\frac{1}{q}}}\right\}^r\right]^m}{m!}$$

$${}_1F_1\left[\frac{-mr}{q}; \frac{\alpha+k}{q}; \frac{qty}{1-qt}\right]$$

$$= \sum_{m=0}^{\infty} \frac{m!}{\left(\frac{\alpha+k}{q}\right)_m} M_m^{(\alpha)}(x, r, p, k, q)\, L_m^{\left(\frac{\alpha+k-q}{q}\right)}(y)\, t^m \tag{7.6}$$

This can be considered as a generalization of the Hille-Hardy formula mentioned above and indeed reduces to it when $p=q=r=1$, $k=0$ and $\alpha=\alpha+1$.

On the other hand, if we taken

$$\alpha_n = \frac{\left(\frac{c}{q}\right)_n}{\left(\frac{\alpha+k}{q}\right)_n}$$

then by (5.11)

$$F[x, t] = e^{px^r}(1-qt)^{-\frac{c}{q}} \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_2F_1\left[-\frac{mr}{q}, \frac{c}{q}; \frac{\alpha+k}{q}; \frac{qt}{qt-1}\right].$$

In this case (7.2) becomes

$$(1-qt)^{\frac{c}{q}-\left(\frac{\alpha+k}{q}\right)}(1-qt+qty)^{-\frac{c}{q}} e^{px^r} \sum_{m=0}^{\infty} \frac{\left[-p\left\{\frac{x}{(1-qt)^{\frac{1}{q}}}\right\}^r\right]^m}{m!}$$

$$\cdot {}_2F_1\left[-\frac{mr}{q}, \frac{c}{q}; \frac{\alpha+k}{q}; \frac{qyt}{1-qt+qty}\right]$$

$$= \sum_{m=0}^{\infty} {}_2F_1\left[-m, \frac{c}{q}; \frac{\alpha+k}{q}; y\right] M_m^{(\alpha)}(x, r, p, k, q)\, t^m. \tag{7.7}$$

Replacing α by $\alpha+1$ and substituting $p=q=r=1$, $k=0$, and after a little simplification one obtains the Weisner's formula.

While concluding we remark that several of our results will reduce to those of Srivastava and Singhal [10] when $k=0$; to those of Chatterjea [3] for $k=0$, $q=1$ and to those of Al-Salam [1] for $p=q=r=k=1$.

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ON SOME PROPERTIES OF A CLASS OF POLYNOMIALS UNIFYING THE GENERALIZED HERMITE LAGUERRE AND BESSEL POLYNOMIALS

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1. Introduction

The aim of Chak [5], who separately studied the two classes of polynomials given by

$$G_{n,k}^{(\alpha)}(x)=x^{-\alpha-kn}\, e^{x}\, \theta^{n}(x^{\alpha}\, e^{-x}) \tag{1.1}$$

and

$$P_{n,r}^{(\alpha)}(x)=\frac{1}{n!}\, x^{-\alpha}\, e^{x^{r}}\, D^{n}\,(x^{n+\alpha}\, e^{-x^{r}}) \tag{1.2}$$

where $\theta \equiv x^{k+1}\, D$, $D\equiv\frac{d}{dx}$, an . that of palas [19] who introduced the polynomials $T_{kn}(x)$ satisfying the Rodrigue's formula

$$T_{kn}(x)=\frac{1}{n!}\, e^{x^{k}}\, D_{u}\left\{x^{n}\, e^{-x^{k}}\right\} \tag{1.3}$$

was to generalize the work of Steffenson [26], Toscano [27], Humbert [13] and Maurice de Duffahe. [10]. In a recent communication [5] we defined the differential operator $T_{k,q}\equiv x^{q}(k+xD)$, q and k are constants, with a view to unify the work of Chak [5], Al Salam [1] and Mittal [18] and employed it to characterize certain classes of polynomials which at the same time provided an extension of the polynomials considered by Srivastava and Singhal [25], who generalized the work of Chatterjea [6] and Singh and Srivastava [24] on generalized Laguerre polynomials. As observed in [16]. our polynomials provide a direct generalization of the Generalized Hermite polynomials of Gould and Hopper [12] whose work is a generalization of the work of researchers like Bell [3], Rajgopal [21] and Riordan [22] on Hermite polynomials Quite naturally, we were led to consider the polynomials

$$\left\{M_{n}^{(\alpha)}(x, r, p, b, k, q) \mid n=0, 1, 2, ..\right\}$$

defined by

$$M_{n}^{(\alpha)}(x, r, p, b, k, q)=c(b, n)\, x^{-\alpha-nq-n}\, e^{px^{r}}\; T_{k,q}^{n}\left(x^{\alpha+bn}\, e^{-px^{r}}\right), \tag{1.4}$$

where $c(b, n)$ is a constant such that

$$c(b, n)=\frac{(-1)^{\frac{n}{2}(b-1)(b-2)}}{(2)^{\frac{nb}{2}(b-1)}\;(1)_{nb(2-b)}},\ b \text{ being a non-negative integer.}$$

This evidently provides us with an elegant unified representation of the various known extensions of the classical Hermite, Laguerre, the Bessel polynomials of Krall & Frank [17] and includes the polynomials due to Singhal and Joshi ([23], see also Chatterjea [7]) as special cases and indeed reduces to them when $k=0$, $q=-1$.

Below, we tabulate some of the obvious connections :

Parameters						M-form	Type and Authors		
k	q	b	p	r	α				
0	-1	0	p	r	α	$M_n^{(\alpha)}(x, r, p, 0, 0, -1)$	$H_n^r(x, \alpha, p)$	Gould and Hopper	[12]
0	1	0	p	r	$\alpha-n+1$	$M_n^{(\alpha-n+1)}(x, r, p, 0, 0, +1)$			
0	-1	1	p	r	α	$M_n^{(\alpha)}(x, r, p, 1, 0, -1)$	$L_n^{(\alpha)}(x, r, p)$	Singh and Srivastava	[24]
0	1	1	p	r	$\alpha-n+1$	$M_n^{(\alpha-n+1)}(x, r, p, 1, 0, 1)$	or $T_{rn}^{(\alpha)}(x, p)$	Chatterjea	[6]
0	-1	2	p	-1	$\alpha-2$	$M_n^{(\alpha-2)}(x, -1, p, 2, 0, -1)$	$\left(\frac{p}{2}\right)^n Y_n(x, \alpha, p)$	Krall and Frink	[17]
0	1	2	p	-1	$\alpha-n-1$	$M_n^{(\alpha-n-1)}(x, -1, p, 2, 0, 1)$			

and cite a few properties of the operator $T_{k,q}$ which will be required in the investigations that follow :

$$T_{k,q}^{n}\left(x^{\alpha+m}\right)=q^{n}\left(\frac{\alpha+m+k}{q}\right)_{n}x^{\alpha+m+nq} \tag{1.5}$$

$$T_{k,q}^{n}\equiv x^{nq}\prod_{j=0}^{n-1}(\delta+k+jq),\ \delta\equiv xD \tag{1.6}$$

$$F(T_{k,q})[x^{\alpha}f(x)]=x^{\alpha}F[T_{k,q}+x^{q}\alpha]f(x) \tag{1.7}$$

$$F(T_{k,q})[e^{g(x)}f(x)]=e^{g(x)}F[T_{k,q}+x^{q+1}g'(x)]f(x) \tag{1.8}$$

$$T_{k,q}^{n}(uv)=\sum_{m=0}^{n}\binom{n}{m}\left(T_{k,q}^{n-m}v\right)\left(T_{q}^{m}u\right),\ T_{q}=x^{q+1}D \tag{1.9}$$

$$e^{tT_{k,q}}\{x^{\alpha}f(x)\}=x^{\alpha}\left(1-x^{q}qt\right)^{-(\alpha+k)/q}f\left(x(1-x^{q}qt)^{-1/q}\right) \tag{1.10}$$

In particular,

$$e^{tT_{k,q}}[x^{\alpha+bn}e^{-px^{r}}]=x^{\alpha+bn}\left(1-x^{q}qt\right)^{-(\alpha+bn+k)/q}\times$$
$$\times\exp[-px^{r}(1-x^{q}qt)^{-r/q}] \tag{1.11}$$

$$T_{k,q}^{m}[x^{\alpha+bn}e^{-px^{r}}]=q^{m}x^{\alpha+bn+mq}\left(\frac{\delta+\alpha+k+bn}{q}\right)_{m}e^{-px^{r}} \tag{1.12}$$

2. The explict form

In view of the definition above and (1.5), we readily obtain

$$M_{n}^{(\alpha)}(x,r,p,b,k,q)=c(b,n)\,q^{n}e^{px^{r}}\sum_{j=0}^{\infty}\frac{(-px^{r})^{j}}{j!}x^{(b-1)n}\left(\frac{\alpha+k+bn+rj}{q}\right)_{n}$$

$$=c(b,n)\,q^{n}\sum_{m=0}^{\infty}\frac{(-p)^{m}}{m!}x^{mr+(b-1)n}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{\alpha+k+bn+rj}{q}\right)_{n} \tag{2.1}$$

Since the inner series represents the m^{th} difference of a polynomial of degree n which vanishes for $m>n$, we can write the explicit formula as

$$M_{n}^{(\alpha)}(x,r,p,b,k,q)=c(b,n)q^{n}x^{(b-1)n}\sum_{m=0}^{n}\frac{(-px^{r})^{m}}{m!}\times$$
$$\times\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\left(\frac{\alpha+k+bn+rj}{q}\right)_{n} \tag{2.2}$$

If

$$\Delta_{\alpha, r} f(\alpha) = f(\alpha+r) - f(\alpha)$$

so that

$$\Delta^{m}_{\alpha,r} f(\alpha) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(\alpha + rj)$$

then it follows that

$$M_n^{(\alpha)}(x, r, p, b, k, q) = c(b, n)\, q^n\, x^{(b-1)n} \sum_{m=0}^{n} \frac{(-px^r)^m}{m\,!} \times$$

$$\times \Delta^{m}_{\alpha+k+bn,\, r} \left(\frac{\alpha+k+bn}{q}\right)_n. \tag{2.3}$$

...hows that these polynomials are of degree n in x^{b+r-1} an...
we have

$$M_n^{(\alpha)}(x, r, \quad . k, q) = c(b, n)\; q^n \exp\left[-px^{b+r-1} \Delta_{\alpha+k+bn, r}\right] \times$$

$$\times \left(\frac{\alpha+k+bn}{q}\right)_n. \tag{2.4}$$

We would like to remark that depending upon $\frac{r}{q}$ as a positive or a negative integer, the explicit formula (2.1) can be transformed into more elegant alternate forms, which may conveniently be termed as the hypergeometric forms. Indeed, since ([20], p. 22)

$$(\alpha)_{nk} = k^{nk} \prod_{j=1}^{k} \left(\frac{\alpha+j-1}{k}\right)_n,$$

then assuming that $\frac{r}{q} = s$, s being a positive integer one gets by an appeal to (1.6)

$$M_n^{(\alpha)}(x, r, p, b, k, q) = q^n c(b, n)\; x^{(b-1)n}\; e^{px^r} \left(\frac{\alpha+k+bn}{q}\right)_n \times$$

$$\times {}_sF_s \left[\begin{matrix} \Delta\left(s, \dfrac{\alpha+k+bn+nq}{q}\right); \\ \Delta\left(s, \dfrac{\alpha+k+bn}{q}\right); \end{matrix} \; -px^r \right], \tag{2.5}$$

and when $\frac{r}{q} = -s$, s being positive integer, we have

$$M_n^{(\alpha)}(x, r, p, b, k, q) = q^n\; c(b, n) x^{(b-1)n}\; e^{px^r} \left(\frac{\alpha+k+bn-nq+q}{q}\right)_n \times$$

$$\times\,{}_sF_s\left[\begin{matrix}\Delta\left(s,\ \dfrac{\alpha+k+bn+q}{q}\right); \\ \Delta\left(s,\ \dfrac{\alpha+k+bn-nq+q}{q}\right);\end{matrix}\ -px^r\right], \tag{2.6}$$

where, as usual, $\Delta(s, \alpha)$ stands for the set of s parameters

$$\frac{\alpha}{s},\ \frac{\alpha+1}{s},\ \ldots,\ \frac{\alpha+s-1}{s}.$$

Formula (2.5) or (2.6) reduces to an elegant result for the polynomials of Srivastava and Singhal [25] which is believed to be new. For instance replacing α by $\alpha-n$, setting $b=1$ and $k=0$ in (2.5), we get

$$G_n^{(\alpha)}(x, r, p, q)=\frac{q^n}{n!}\left(\frac{\alpha}{q}\right)_n e^{px^r}\,{}_sF_s\left[\begin{matrix}\Delta\left(s,\ \dfrac{\alpha+nq}{q}\right); \\ \Delta\left(s,\ \dfrac{\alpha}{q}\right);\end{matrix}\ -px^r\right] \tag{2.7}$$

Note also that, on specializing the parameters in (2.5), one would obtain formulas for the various polynomials set described in the introduction. Of particular interest, when $b=2$, $k=0$, $q=r=-1$ and α is replaced by $\alpha-2$, is the explicit formula for the generalized Bessel polynomials of Krall & Frink [17] which does not seem to have been noticed earlier, viz.

$$Y_n(x, \alpha, p)=\left(\frac{x}{p}\right)^n(\alpha+n-1)_n\ {}_1F_1\left[-n;\ 2-\alpha-2n;\ \frac{p}{x}\right] \tag{2.8}$$

But it has been shown by Chatterjea [8] that

$$Y_n(x, \alpha, p)={}_2F_0\left[-n,\ \alpha+n-1;\ -;\ -\frac{x}{p}\right]. \tag{2.9}$$

A comparison of (2.8) and (2.9), would thus lead us to the transformation formula

$${}_2F_0\left[-n,\ \alpha+n-1;\ -;\ -\frac{x}{p}\right]=\left(\frac{x}{p}\right)^n(\alpha+n-1)_n\times$$
$$\times\,{}_1F_1\left[-n;\ 2-\alpha-2n;\ \frac{p}{x}\right] \tag{2.10}$$

3. The generating relations.

From (2.1), it follows that

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{c(b, n)}=e^{px^r}\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m!}\times$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left(\frac{mr+\alpha+k}{q}\right)_{n+\frac{bn}{q}}}{\left(\frac{mr+\alpha+k}{q}\right)_{\frac{bn}{q}}} (qtx^{b-1})^n. \tag{3.1}$$

Therefore, assuming that $\frac{b}{q}=h$, h being a positive integer, we are led to the generating relation

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{c(b, n)} = e^p \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \times$$

$$\times {}_{h+1}F_h \left[\begin{matrix} \triangle\left(h+1, \frac{mr+\alpha+k}{q}\right); \\ \triangle\left(h, \frac{mr+\alpha+k}{q}\right); \end{matrix} \; \frac{(h+1)^{h+1} qtx^{b-1}}{h^h} \right] \tag{3.2}$$

Note from Bailey [2] that

$$F_{h-1}\left[\begin{matrix}\triangle(h, d); \\ \triangle(h-1, d);\end{matrix} -\frac{xh^h}{(h-1)^{h-1}(1-x)^h}\right] = \frac{(1-x)^d}{1+(h-1)x} \tag{3.3}$$

and since

$$\frac{x}{(1-x)^h} = \frac{x}{1-x}\left(1+\frac{x}{1-x}\right)^h \tag{3.4}$$

$$\frac{(1-x)^d}{1+(h-1)x} = \frac{\left[1+\frac{x}{1-x}\right]^{1-d}}{1+\frac{hx}{1-x}}; \tag{3.5}$$

we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{M_n^{(\alpha)}(x, r, p, b, k, q)}{c(b, n)} = (1+\xi)^{-(\alpha+k+q)/q} \times$$

$$\times \{1+(h+1)\xi\}^{-1} \exp\left[px^r\{1-(1+\xi)^{-r/q}\}\right] \tag{3.6}$$

where ξ is given by $\xi(1+\xi)^h = -qtx^{b-1}$.

When $p=q=r=k=1$ so that $h=1$ and $\xi(1+\xi)=-t$, we have, for the generalized Laguerre polynomials [1], a new generating relation in the form

$$\sum_{n=0}^{\infty} t^n L_n^{(\alpha+n)}(x)=[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-4t)}]^{\alpha+2}[1-4t]^{-1/2}\times$$

$$\times\exp\,[x\{1-(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-4t)})^{-1}\}] \tag{3.7}$$

On the other hand, if $\frac{b}{q}=-h$, h being a positive integer, where q is essentially a negative integer and b takes positive values only, the formula (3.1) admits the form

$$\sum_{n=0}^{\infty}\frac{t^n}{n\,!}\,\frac{M_n^{(\alpha)}(x,r,p,b,k,q)}{c(b,n)}=e^{px^r}\sum_{m=0}^{\infty}\frac{(-px^r)^m}{m\,!}\times$$

$$\times\,{}_hF_{h-1}\left[\begin{matrix}\Delta\left(h,\dfrac{mr+\alpha+k+q}{q}\right); \\ \Delta\left(h-1,\dfrac{mr+\alpha+k+q}{q}\right);\end{matrix}\ \frac{h^h q t x^{b-1}}{(h-1)^{h-1}}\right] \tag{3.8}$$

Thus, in view of the relations (3.3)–(3.5), we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\frac{M_n^{(\alpha)}(x,r,p,b,k,q)}{c(b,n)}=(1+\xi)^{-(\alpha+k)/q}(1+h\xi)^{-1}\times$$

$$\times\exp\,[px^r\{1-(1+\xi)^{-r/q}\}] \tag{3.9}$$

where $\qquad \xi\,(1+\xi)^{h-1}=qtx^{b-1}.$

Replacing α by $\alpha-2$, t by $\frac{t}{2}$ and setting $b=2$, $q=1$, $k=0$, and $r=-1$, so that $h=2$ and $\xi(1+\xi)=-tx$, (3.9) yields the known relation ([4], see also [9]),

$$\sum_{n=0}^{\infty}\frac{(pt/2)^n}{n\,!}\,Y_n(x,\alpha,p)=(1-2xt)^{-1/2}\{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-2xt)}\}^{2-\alpha}\times$$

$$\times\exp\left[\frac{p}{2x}\left\{1-\sqrt{(1-2xt)}\right\}\right] \tag{3.10}$$

for the generalized Bessel polynomials of Krall and Frink.

Formulas (3.3) and (3.9) admit further generalizations. However, generalizations of (3.9) are omitted here for reasons of brevity and of similarity.

For (3.3), proceeding in a manner similar to that of (3.2), it is readily established that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n M_n^{(\alpha)}(x, r, p, b, k, q)t^n}{n!\, c(b, n)\, (\rho)_n} = \exp(px^r) \times \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \times$$

$$\times {}_{h+2}F_{h+1} \left[\begin{matrix} \lambda, \Delta\left(h+1, \dfrac{mr+\alpha+k}{q}\right); \\ \rho, \Delta\left(h, \dfrac{mr+\alpha+k}{q}\right); \end{matrix} \quad \frac{(h+1)^{h+1} qtx^{b-1}}{h^h} \right] \quad (3.11)$$

This suggests the existence of a more general result in the form

$$\sum_{n=0}^{\infty} \frac{[(\lambda_\mu)]_n M_n^{(\alpha)}(x, r, p, b, k, q)t^n}{[(\rho_\nu)]_n\, n!\, c(b, n)} = \exp(px^r) \times \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \times$$

$$\times {}_{\mu+h+1}F_{\nu+h} \left[\begin{matrix} (\lambda_\mu), \Delta\left(h+1, \dfrac{mr+\alpha+k}{q}\right); \\ (\rho_\nu), \Delta\left(h, \dfrac{mr+\alpha+k}{q}\right); \end{matrix} \quad \frac{(h+1)^{h+1} qtx^{b-1}}{h^h} \right] \quad (3.12)$$

which can also be proved analogously.

It will be understood throughout that (λ_μ) stands for the set of μ-parameters $\lambda_1, \lambda_2, \ldots, \lambda_\mu$. Similarly, (ρ_ν) stands for the set of ν parameters $\rho_1, \rho_2, \ldots, \rho_\nu$; and that $[(\lambda_\mu)]_n$ has the interpretation $\prod_{j=1}^{\mu} (\lambda_j)_n$, etc.

Next, consider the identity

$$e^{tT_{k,q}} [x^{\alpha+bn} e^{-px^r}] = \sum_{m=0}^{\infty} \frac{t^m}{m!} T_{k,q}^m (x^{\alpha+bn} e^{-px^r}).$$

Making an appeal to (1.11) and by a simple change in the parameter one obtains

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{M_m^{(\alpha-bm)}(x, r, p, b, k, q)}{c(b, m)} = (1-qtx^{b-1})^{-(\alpha+k)/q} \times$$

$$\times \exp[px^r\{1-(1-qtx^{b-1})^{-r/q}\}] \quad (3.13)$$

For $b=2$, $k=0$, $q=1$, $r=-1$ and replacing α by $\alpha-1$,

$$\sum_{m=0}^{\infty} \frac{(pt)^m}{m!} Y_m(x, \alpha-m, p) = (1-xt)^{1-\alpha} e^{pt}, \quad (3.14)$$

which is due to Chatterjea [9].

Following the method of proof outlined for (3.2), formula (3.11) can be further extended to yield

$$\sum_{n=0}^{\infty} \frac{[(\lambda_\mu)]_n \, M_n^{(\alpha-bn)}(x, r, p, b, k, q)t^n}{[(\rho_\nu)]_n \, c(b, n)\, n!} = \exp(px^r) \times \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \times$$

$$\times {}_{\mu+1}F_\nu \left[\begin{matrix} (\lambda_\mu), \dfrac{mr+\lambda+k}{q} ; \\ (\rho_\nu) ; \end{matrix} \; qtx^{b-1} \right] \tag{3.15}$$

which evidently corresponds to (3.13), when $\lambda_\mu = \rho_\nu = 1$.

A typical deduction from (3.15) is the formula

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n \, M_n^{(\alpha-bn)}(x, r, p, b, k, q)\, t^n}{n!\, c(b, n) \left(\dfrac{\alpha+k}{q}\right)_n} = \exp(px^r)(1-qtx^{b-1})^{-\lambda} \times$$

$$\times \sum_{m=0}^{\infty} \frac{(-px^r)^m}{m!} \, {}_2F_1 \left[\begin{matrix} \lambda, \dfrac{mr}{q} ; \\ \dfrac{\alpha+k}{q} ; \end{matrix} \; \frac{qtx^{b-1}}{1-qtx^{b-1}} \right] \tag{3.16}$$

and this reduces to (3.a.10) of Joshi [14] when $k=0$, $q=-1$ and $\lambda=-\alpha$.

4. A bilinear genarating function.

Theorem. *If*

$$F[x, t] = \sum_{m=0}^{\infty} \frac{a_m t^m x^{(1-b)m}}{m!\, c(b, m)} M_m^{(\alpha-bm)}(x, r, p, b, k, q) \tag{4.1}$$

then

$$(1-qtx^{b-1})^{\frac{-(\alpha+k)}{q}} \exp[px^r\{1-(1-qtx^{b-1})^{-r/q}\}] F\left[\frac{x}{(1-qtx^{b-1})^{1/q}}, \frac{x^{b-1}yt}{1-qtx^{b-1}}\right]$$

$$\sum_{m=0}^{\infty} \gamma_m(y) \frac{t^m}{m!\, c(b, m)} M_m^{(\alpha-bm)}(x, r, p, b, k, q) \tag{4.2}$$

where

$$\gamma_m(y) = \sum_{s=0} \binom{m}{s} a_s y^s \tag{4.3}$$

Proof. Let

$$F[x,t]=\sum_{m=0}^{\infty}\frac{a_m t^m x^{(1-b)m}}{m!\,c(b,m)}M_m^{(\alpha-bm)}(x,r,p,b,k,q)$$

be a given generating function for $M_m^{(\alpha-bm)}(x,r,p,b,k,q)$. Replacing t by $tx^q y$, multiplying both sides by $x^\alpha e^{-px^r}$ and then operating by $e^{tT_{k,q}}$ and making an appeal to (1.10), the left hand side equals

$$x^\alpha(1-x^q qt)^{\frac{-(\alpha+k)}{q}}\exp[-px^r(1-x^q qt)^{-r/q}]F\left[\frac{x}{(1-x^q qt)^{1/q}},\frac{x^q yt}{1-x^q qt}\right].$$

On the other hand, the right hand side

$$=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\frac{t^s\,T_{k,q}^s\,a_m t^m y^m x^{mq}x^{(1-b)m}}{s!\,m!\,c(b,m)}\cdot x^\alpha e^{-px^r}M_m^{(\alpha-bm)}(x,r,p,b,k,q)$$

$$=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\frac{t^{m+s}a_m y^m}{m!\,s!}T_{k,q}^{m+s}\left(x^\alpha e^{-px^r}\right)$$

$$=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\frac{t^m a_{m-s}y^{m-s}}{s!\,(m-s)!}T_{k,q}^{m}\left(x^\alpha e^{-px^r}\right)$$

$$=\sum_{m=0}^{\infty}\frac{t^m}{m!}T_{k,q}^{m}\left(x^\alpha e^{-px^r}\right)\sum_{s=0}^{m}\binom{m}{s}a_{m-s}y^{m-s}$$

$$=x^\alpha e^{-px^r}\sum_{m=0}^{\infty}\frac{(tx^{q-b+1})^m}{m!\,c(b,m)}M_m^{(\alpha-bm)}(x,r,p,b,k,q)\,\gamma_m(y).$$

Equating the above with the left hand side replacing t by $\frac{t}{x^{q-b+1}}$, we obtain the desired result.

Some of the immediate corollaries of the above theorem are the bilinear generating relations for the generalized Hermite, Laguerre and Bessel polynomials which are believed to be new.

Thus:

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